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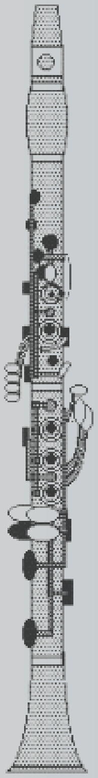
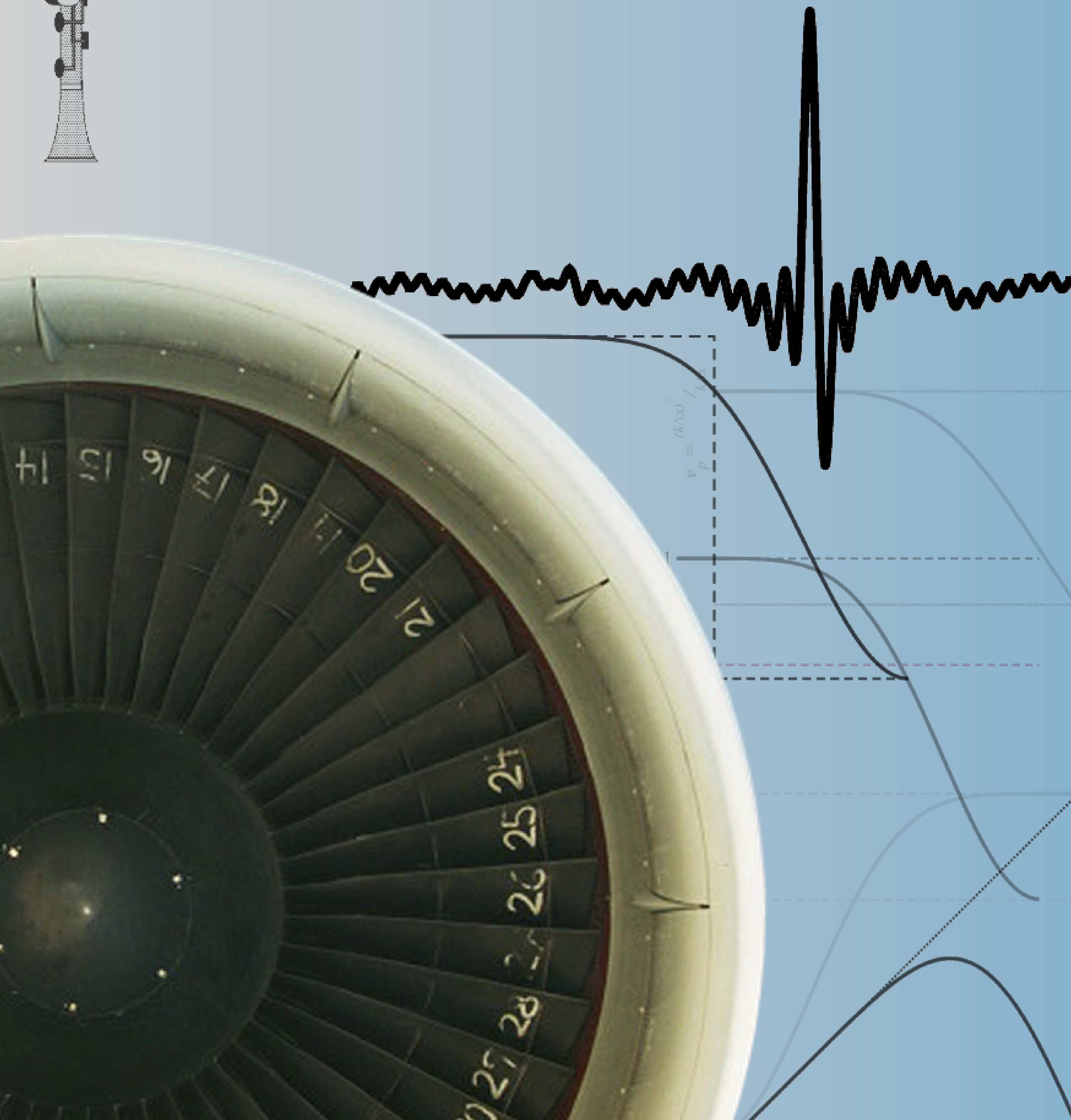


Computational AeroAcoustics
and it's Applications



Contents

1 Introduction to Computational Fluid Dynamics (CFD) and Computational Aeroacoustics (CAA)

1.1 Fluid Dynamics

Fluid Dynamics is a branch of classical physics. It studies the fluid-flow phenomena, nature and the conservation laws of classical physics.
Some related topics:

- Jets
- Turbulence
- Rotating Flows
- Shear and Boundary Layers
- Aeroacoustics
- Bluff Body Flows

1.2 Acoustics

Acoustics is the science of sound. It studies the sound generation, transmissions and effects.
Some related technical fields:

- Acoustical Oceanography
- Animal Bio acoustics
- Architectural Acoustics
- Biomedical Acoustics
- Engineering Acoustics
- Musical Acoustics
- Noise
- Physical Acoustics
- Psychological and Physiological Acoustics
- Signal Processing in Acoustics
- Speech Communication
- Structural Acoustics and Vibration
- Underwater Acoustics

1.3 Approaches

- Theoretical
- Experimental (EFD – Experimental Fluid Dynamics)
- Computational (CFD – Computational Fluid Dynamics)

1.4 Experimental vs. Computational Fluid Dynamics

Objective: To achieve a quantitative description of the fluid flow phenomena.

EFD

- Experimental facilities including all the necessary measurement techniques
- Slow and Expensive
- Sequential
- Limited number of points in space and time
- Expensive to change geometric parameters
- Measurement errors

CFD

- Computers, mathematical models, numerical methods and software
- Faster and Cheaper
- Parallel
- Higher resolution in space and time
- Easy to change geometric parameters
- Numerical errors: modeling, numerical method and implementation

1.5 Definition of CFD

CFD is a branch of applied mathematics and it is the *art* of replacing the differential equation governing the Fluid Flow with a set of algebraic equations (the process is called discretization), which in turn can be solved with the aid of a digital computer to get an *approximate* solution. The well known discretization methods used in CFD are Finite Difference Method (FDM), Finite Volume Method (FVM), Finite Element Method (FEM) and Boundary Element Method (BEM).

FDM is the oldest (Euler) method in CFD applications. Here the domain including the boundary of the physical problem is covered by a *grid* or *mesh*. At each of the interior grid points the original differential equations are replaced by equivalent finite difference approximations. In making this replacement, we introduce an error that is proportional to the size of the grid. An accurate solution within a specified tolerance can be achieved by decreasing the error through decreasing the grid size.

FVM is a numerical method for solving partial differential equations that calculates the values of the conserved variables averaged across a fixed region in space referred as a *control volume*. One advantage of the finite volume method over finite difference methods is that it does not require a structured mesh (although a structured mesh can also be used). Furthermore, the finite volume method is preferable to other methods as a result of the fact that boundary conditions can be applied non invasively. This is true because the values of the conserved variables are located *within* the volume element, and not at nodes or surfaces. Finite volume methods are especially powerful on coarse nonuniform grids and in calculations where the mesh moves to track interfaces or shocks.

FEM is a mathematical (numerical) tool (just like Finite Difference Method) used to solve complex physical problems which are not amenable to classical techniques of mathematics. It has found it's applications in the fields of Structural Design, Vibration Analysis, Fluid Dynamics, and Heat Transfer to name a few.

The basic idea in FEM analysis of field problems is as follows:

- The solution domain is discretized into a number of small sub-regions (i.e. Finite Elements).
- Select an approximating function known as interpolation polynomial to represent the variation of the dependent variable over the elements.
- Integration of the governing differential equation (often several) with a suitable Weighting Function over each element to produce a set of algebraic equations - one equation for each element.
- The set of algebraic equations are then solved to get the *approximate* solution of the problem.

In principle, any well-posed Boundary Value Problem can be solved by the techniques of FEM.

BEM is an important technique in the computational solution of a number of physical or engineering problems. It is essentially a method for solving partial differential equations. The boundary element method has the important distinction that only the boundary of the domain of interest requires discretization. In the BEM, only the boundary is discretized. Hence, the mesh generation is considerably simpler for this method than for the volume methods. The boundary element method transforms the differential operator defined in the domain to integral operators defined on the boundary. Boundary solutions are obtained directly by solving the set of linear equations. However, potentials and gradients in the domain can be evaluated only after the boundary solutions have been obtained.

1.6 Important Factors of CFD

1. Physics	The physics of fluid flow is governed by the partial differential equations. In the field of aerodynamics, the governing equations are the compressible Navier-Stokes Equation. *In general* these are non-linear and no analytical solutions exist for them.
2. Modeling	If the viscosity of the fluid-flow is unimportant ($Re \to \infty$) then the governing equations can be reduced to the Euler Equations. For particular physical phenomenon, the governing equations can be simplified.

3.	Numerics	Grid generation, discretization of the PDE, boundary conditions, solve a system of algebraic equations. Truncation Errors generated by numerical procedures.
4.	Visualization	
5.	Verification / Validation	V&V

1.7 Definition of Computational AeroAcoustics (CAA)

CAA is a sub-discipline of CFD, it has its own objectives, characteristics, issues and methods.

Objective: To understand the physics of noise/sound generation and propagation, e.g. for community noise prediction and aircraft certification.

Special Characteristics and Issues:

- Unsteady → Time Domain method
- Long Propagation Distance → FW/H, Kirchhoff
- Large Spectral Bandwidth → Time Domain method
- Radiation and Outflow Boundary Conditions
- Solid Wall and Impedance Boundary Conditions
- Acoustic Wave and Mean Flow Disparity
- Nonlinearity

Requirements on Numerical Schemes:

- Minimum dissipation and dispersion errors ⇔ correct prediction on both amplitude and phase
- Special bounding conditions e.g.
  - Radiation
  - Outflow
  - PML (Perfect Matched Layer)
- Less PPW (Points Per Wave length) possible

1.8 Classification of Fluid Flows/Acoustic Problems

1. The solution region of the problem
2. The nature of the equation describing the problem or
3. The associated boundary conditions

1.8.1 Classification of Solution Regions

Interior problem: Inner, closed or bounded problem.
(e.g. wave propagation inside a duct)

Exterior problem: Outer, open or unbounded problem.
(e.g. radiation from an oscillating sphere)

1.8.2 Classification of the Nature of the Equations

Mathematical Classification (Classification of Differential Equations)

- Elliptic
- Parabolic
- Hyperbolic

Physical Classification

Viscous	Inviscid
Compressible	Incompressible
Equilibrium Problem (steady)	Marching problems (transient)
Laminar	Turbulence
Single-Phase	Multiphase

1.8.3 Classification of Boundary Conditions

Dirichlet boundary condition	$\Phi(\vec{r}) = p(\vec{r})$, $\vec{r}$ on S
Neumann boundary condition	$\partial\Phi(\vec{r})/\partial n = q(\vec{r})$
Mixed boundary condition	$\partial\Phi(\vec{r})/\partial n + h(\vec{r})\,\Phi(\vec{r}) = w(\vec{r})$

where $p(\vec{r})$, $q(\vec{r})$, $h(\vec{r})$ and $w(\vec{r})$ are explicitly known functions on the boundary S and Φ is a general value (e.g. pressure, density, ...).

1.9 Some CFD and CAA Applications

Figure 1.1: Aerodynamic noise

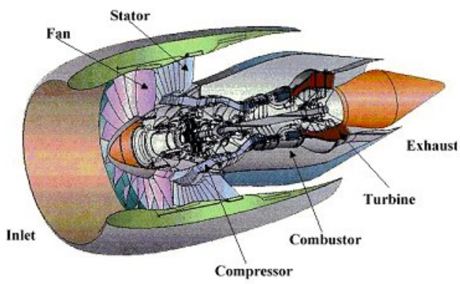


Figure 1.2: Engine noise

Figure 1.3: Architectural acoustics

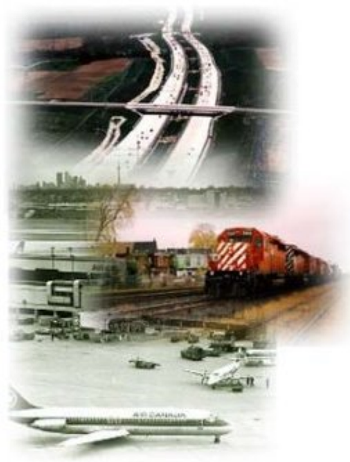


Figure 1.4: Environmental noise and vibration

Figure 1.5: Underwater acoustics

Figure 1.6: Music acoustics

Figure 1.7: Medical acoustics

Figure 1.8: Industrial acoustics

1.10 Zonal approach / Example: engine noise computation

To compute the distribution of noise produced by an engine, splitting the region in different zones is a common procedure (zonal approach). Different numerical techniques are used.

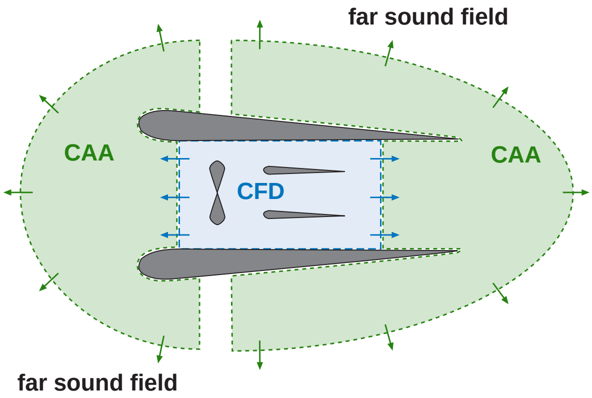


Figure 1.9: Zonal approach to compute engine noise

Inside the engine many effects occur: turbulence, heat, unsteady flow, rotating boundaries. To obtain the sources of sound inside the engine, **CFD** with a fine mesh is used to solve the Navier Stokes equation as accurate as possible. Some methods are:

- Unsteady Reynolds Averaged Navier Stokes (URANS)
- Large Eddy Simulation (LES)
- Direct Numerical Simulaton (DNS)

To calculate the sound propagation outside the engine, where still boundaries exist and the mean flow is not constant, the calculated sources of sound are used for the **CAA** methods. A coarser mesh is sufficient to solve the linearized Euler equation. Some methods are:

- Finite Element Method (FEM), spectral elements
- Discontinious Galerkin
- Arbitrary high order schemes using DERivatives (ADER)
- Finite Difference (FD)

The **far field** with an almost constant mean flow and without further boundaries can be calculated by using the well known far field approximations. Some methods are:

- Equivalent Source Method (ESM) (multipol expansion inside CAA / CFD region)
- Boundary Element Method (BEM) (Kirchhoff: poles on boundary)
- Lighthills analogy (equation of Ffowcs-Williams & Hawkings)

1.11 Mathematical Classification

The conservation principles are expressed as the Partial Differential Equations (PDEs). The PDEs are therefore at the foundation of computational science. A wide variety of PDEs are encountered in the study of physical phenonema. The PDEs can be classified according to mathematical features or according to the type of physical phenonema involved. It is critical to classify the PDEs since the solution methods depend on the structure of the equation.

The classical mathematical classification of elementary PDEs sterns from the analysis of the general second order PDE

$$A\Phi_{xx} + B\Phi_{xy} + C\Phi_{yy} + D\Phi_x + E\Phi_y + F\Phi + G = 0.$$

The equations of the characteristics in physical space are as follows. The nature of the equation is therefore determined by the coefficients, according to the following classification:

$\left(\frac{dy}{dx}\right)_{1,2} = \frac{B\pm\sqrt{B^2-4AC}}{2A}$	Elliptic	$B^2 - 4AC < 0$
	Parabolic	$B^2 - 4AC = 0$
	Hyperbolic	$B^2 - 4AC > 0$

Examples of the three types of the PDEs:

Elliptic	→	Poisson's equation	$\nabla^2 u = -\rho/c$
Parabolic	→	Diffusion equation	$\partial u/\partial t = a\partial^2 u/\partial x^2$
Hyperbolic	→	Wave equation	$\partial^2 u/\partial t^2 = a^2\partial^2 u/\partial x^2$

1.12 Definition of Characteristics

Characteristics are lines (2D) or surfaces (3D) along which certain properties remain constant or certain derivatives may be discontinuous. In case of the one-dimensional pressure wave equation a general solution is

$$p'(x,t) = f(x - ct) + g(x + ct)$$

where c is the speed of sound, f is a common plane wave in positive x-direction and g in negative x-direction. If the wave only propagates in positive x-direction the pressure is constant along its characteristic $\xi = x - ct = const$ and therefore the solution proceeds along this characteristic.

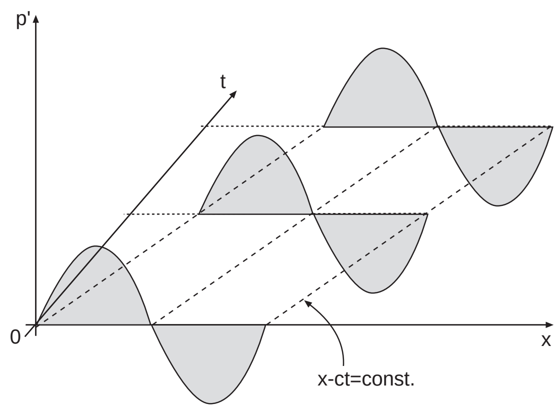


Figure 1.10: Characteristic of a plane wave

The characteristics, if they exist and are real curves within the solution domain, represent the locus of points along which the second derivative may not be continuous.

The general second order PDE

$$A\Phi_{xx} + B\Phi_{xy} + C\Phi_{yy} + D\Phi_x + E\Phi_y + F\Phi + G = 0$$

and the differentials

$$\begin{aligned} d\Phi_x &= \Phi_{xx}dx + \Phi_{xy}dy \\ d\Phi_y &= \Phi_{yx}dx + \Phi_{yy}dy \end{aligned}$$

can be set to a linear system of equations

$$\underbrace{\begin{bmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{bmatrix}}_{\text{coefficient matrix}} \cdot \begin{bmatrix} \Phi_{xx} \\ \Phi_{xy} \\ \Phi_{yy} \end{bmatrix} = \begin{bmatrix} -(D\Phi_x + E\Phi_y + F\Phi + G) \\ d\Phi_x \\ d\Phi_y \end{bmatrix}$$

If the determinant of the coefficient matrix is zero, then there may be no unique solution for the second derivatives (discontinuity). Using the rule of Sarrus for the coefficient matrix leads to

$$\begin{aligned} Adydy + Cdxdx - Bdxdy &= 0 \\ A\left(\frac{dy}{dx}\right)^2 - B\frac{dy}{dx} + C &= 0 \\ \left(\frac{dy}{dx}\right)^2 - \frac{B}{A}\frac{dy}{dx} + \frac{C}{A} &= 0. \end{aligned}$$

The solution of this quadratic equation is given in the box above to determine the mathematical classification (elliptic, parabolic or hyperbolic PDE).

1.13 The Well-Posed Problem

In order for a problem involving a PDE to be well-posed

1. the solution to the problem must exist,
2. the solution to the problem must be unique and
3. the solution to the problem must depend continuously upon the initial or boundary data.

Example 1

Demonstrate the problem of continuous dependence on boundary data.

Laplace's equation:

$$U_{xx} + U_{yy} = 0 \quad -\infty < x < \infty \quad y \geq 0$$

Boundary Condition:

$$\begin{aligned} U(x,0) &= 0 \\ U_y(x,0) &= \frac{1}{n}\sin(nx) \qquad n > 0 \end{aligned}$$

Using separation of variables, we obtain

$$U = \frac{1}{n^2}\sin(nx)\sinh(ny)$$

Analysis:
For large n, we have

$$\begin{aligned} U &\sim \frac{1}{n^2}e^{ny} \\ U_y &\sim \frac{1}{n}e^{ny} \end{aligned}$$

However, from the boundary condition, we have

$$\begin{aligned} U(x,0) &= 0 \\ U_y(x,0) &= \frac{1}{n}\sin(nx) \end{aligned}$$

By comparing the behaviors of $U$and U_y from the analysis of the solution and the boundary conditions, we can easily see that the continuity with the initial data is lost. Therefore the problem is a ill-posed problem.

Since the Laplace's equation is elliptic type, the solution depends on conditions on the entire boundary of the closed domain. In the above example, the boundary conditions are only given on the line $y = 0$. This caused the problem to be ill-posed.

The correct boundary conditions (e.g.) should be

$$\begin{aligned} x &= 0, \qquad U = U_1 \\ x &= L, \qquad U = U_2 \\ y &= 0, \qquad U = U_3 \\ y &= H, \qquad U = U_4 \end{aligned}$$

Example 2

Solve the second-order wave equation in characteristic coordinates

$$U_{\xi\eta} = 0, \qquad \begin{aligned} \xi &= x + ct \\ \eta &= x - ct \end{aligned}$$

Initial data

$$\begin{aligned} U(0,\eta) &= \varphi(\eta) \\ U_\xi(0,\eta) &= \psi(\eta) \end{aligned}$$

Taylor-series expansion in ξ to obtain

$$U(\xi,\eta) = U(0,\eta) + \xi U_\xi(0,\eta) + \frac{\xi^2}{2} U_{\xi\xi}(0,\eta) + \ldots$$

We have (due to $U_{\xi\eta}(0,\eta) = 0$)

$$\psi(\eta) = \text{constant} = c_1$$

and (due to the permutability of the second order derivatives and $U_{\xi\eta} = 0$)

$$\begin{aligned} \frac{\partial U_{\xi\eta}}{\partial \xi} &= \frac{\partial U_{\xi\xi}}{\partial \eta} = 0 \\ \Rightarrow U_{\xi\xi} &= f(\xi) \\ \Rightarrow U_{\xi\xi}(0,\eta) &= \text{constant} = c_2 \end{aligned}$$

Putting altogether in the Taylor-series expansion we obtain

$$\begin{aligned} U(\xi,\eta) &= \varphi(\eta) + c_1\xi + \frac{\xi^2}{2} c_2 \\ &= \varphi(\eta) + g(\xi) \end{aligned}$$

We are not able to uniquely determine the function $g(\xi)$ when the initial data one given along the characteristic $\xi = 0$. The problem is ill-posed.

2 A Review of Finite Difference Methods

2.1 Definitions

Consistency: A finite difference approximation of a partial difference equation (PDE) is consistent if the finite difference equation (FDE) approaches the PDE as the grid size (Δx, Δy, Δz) approaches zero. FDE $\Rightarrow$ PDE.

Stability: If **errors** (truncation, round-off, mistakes) from any source don't grow as the calculation proceeds from one marching step to the other.

Convergence: A finite difference scheme is convergent if the **solution** of the FDE approaches that of the PDE as the grid size (Δx, Δy, Δz) approaches zero.

Lax's equivalence theorem:
For a well-posed initial value FDE, stability and consistency of the FDE are the necessary and sufficient condition for convergence.

2.2 Taylor Series Expansion

2.2.1 Examples

Given an analytical function $f(x)$, the values $f(x_l + \Delta x)$ and $f(x_l - \Delta x)$ can be expanded in a Taylor series about x_l as

$$\begin{aligned} f_{l+1} = f(x_l + \Delta x) = & f_l + \left(\frac{\partial f}{\partial x}\right)_l \Delta x + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}\right)_l \Delta x^2 \\ & + \frac{1}{3!}\left(\frac{\partial^3 f}{\partial x^3}\right)_l \Delta x^3 + \dots \end{aligned} \tag{2.1}$$

respectively

$$\begin{aligned} f_{l-1} = f(x_l - \Delta x) = & f_l - \left(\frac{\partial f}{\partial x}\right)_l \Delta x + \frac{1}{2!}\left(\frac{\partial^2 f}{\partial x^2}\right)_l \Delta x^2 \\ & - \frac{1}{3!}\left(\frac{\partial^3 f}{\partial x^3}\right)_l \Delta x^3 + \dots \end{aligned} \tag{2.2}$$

Eq. (2.1) - Eq. (2.2) results in

$$\left(\frac{\partial f}{\partial x}\right)_l = \frac{f_{l+1} - f_{l-1}}{2\Delta x} + \mathcal{O}\left(\Delta x^2\right)$$

This is known as a three-point-stencil central difference approximation of order $\mathcal{O}(\Delta x)^2$. For higher order accuracies, such as $\mathcal{O}(\Delta x^4)$ and $\mathcal{O}(\Delta x^6)$, the first order derivative, $\left(\frac{\partial f}{\partial x}\right)_l$, can be approximated with additional positions as

$$\left(\frac{\partial f}{\partial x}\right)_l = \frac{-f_{l+2} + 8f_{l+1} - 8f_{l-1} + f_{l-2}}{12\Delta x} + \mathcal{O}\left(\Delta x^4\right) \qquad \text{5-point fourth-order stencil}$$

$$\left(\frac{\partial f}{\partial x}\right)_l = \frac{f_{l+3} - 9f_{l+2} + 45f_{l+1} - 45f_{l-1} + 9f_{l-2} - f_{l-3}}{60\Delta x} + \mathcal{O}\left(\Delta x^6\right) \qquad \text{7-point sixth-order stencil}$$

For higher order derivatives such as $\frac{\partial^2 f}{\partial x^2}$ and $\frac{\partial^3 f}{\partial x^3}$, we can derive

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_l = \frac{f_{l+1} - 2f_l + f_{l-1}}{(\Delta x)^2} + \mathcal{O}\left(\Delta x^2\right)$$

and

$$\left(\frac{\partial^3 f}{\partial x^3}\right)_l = \frac{f_{i+2} - 2f_{i+1} + 2f_{i-1} - f_{i-2}}{2(\Delta x)^3} + \mathcal{O}\left(\Delta x^2\right)$$

2.2.2 First Derivative

In general, consider the approximation of the first order spatial derivative $\frac{\partial f}{\partial x}$ by a finite difference formulation on a uniform grid of spacing Δx

$$\left(\frac{\partial f}{\partial x}\right)_l = \frac{1}{\Delta x}\sum_{j=-N}^{M} a_j f_{l+j} + \mathcal{O}\left(\Delta x^{N+M}\right) \tag{2.3}$$

where a_j are coefficients to be determined and l is an integer representing a grid point. The total points of stencils used are $(N + M + 1)$-point-stencil.

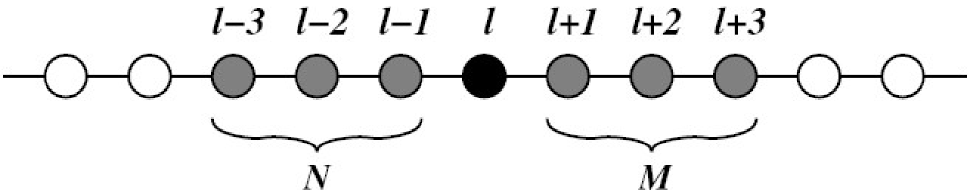


Figure 2.1: Nomenclature of the stencil

Using the Taylor series expansion, we have

$$\begin{aligned}\left(\frac{\partial f}{\partial x}\right)_l =& \frac{1}{\Delta x}\Bigg[a_{-N}\underbrace{\left(f(x_l) + \left(\frac{\partial f}{\partial x}\right)_l(-N\Delta x) + \left(\frac{\partial^2 f}{\partial x^2}\right)_l\frac{1}{2!}(-N\Delta x)^2 + \ldots\right)}_{f_{l-N}} \\
&+ a_{-N+1}\underbrace{\left(f(x_l) + \left(\frac{\partial f}{\partial x}\right)_l(-N+1)\Delta x + \left(\frac{\partial^2 f}{\partial x^2}\right)_l\frac{1}{2!}\left[(-N+1)\Delta x\right]^2 + \ldots\right)}_{f_{l-N+1}} \\
&+ \ldots \\
&+ a_0 f_l \\
&+ \ldots \\
&+ a_{M-1}\underbrace{\left(f(x_l) + \left(\frac{\partial f}{\partial x}\right)_l(M-1)\Delta x + \left(\frac{\partial^2 f}{\partial x^2}\right)_l\frac{1}{2!}\left[(M-1)\Delta x\right]^2 + \ldots\right)}_{f_{l+M-1}} \\
&+a_M\underbrace{\left(f(x_l) + \left(\frac{\partial f}{\partial x}\right)_l M\Delta x + \left(\frac{\partial^2 f}{\partial x^2}\right)_l\frac{1}{2!}(M\Delta x)^2 + \ldots\right)}_{f_{l+M}}\Bigg] + \mathcal{O}\left(\Delta x^{N+M}\right)\end{aligned}$$

Gathering terms of same order in derivation and taking $\frac{1}{\Delta x}$ out we get

$$\left(\frac{\partial f}{\partial x}\right)_l = \frac{1}{\Delta x}\left[f\left(x_l\right)\sum_{j=-N}^{M} a_j + \Delta x\left(\frac{\partial f}{\partial x}\right)_l \sum_{j=-N}^{M} a_j j \right.$$

$$+\frac{1}{2!}\left(\Delta x\right)^2\left(\frac{\partial^2 f}{\partial x^2}\right)_l \sum_{j=-N}^{M} a_j j^2 + \ldots \tag{2.4}$$

$$\left. +\frac{1}{(N+M)!}\left(\Delta x\right)^{N+M}\left(\frac{\partial^{(N+M)} f}{\partial x^{(N+M)}}\right)_l \sum_{j=-N}^{M} a_j j^{N+M}\right] + \mathcal{O}\left(\Delta x^{N+M}\right)$$

Comparing the left hand side with the right hand side of the equation we obtain the

coefficients for first derivative

$$\sum_{j=-N}^{M} a_j = 0 \quad \overset{from}{\longrightarrow} \quad f(x_L)\left(\sum_j a_j\right) \quad = \tfrac{\partial f}{\partial x} + 0\cdot f(x_L)$$

$$\sum_{j=-N}^{M} a_j j = 1 \quad \overset{from}{\longrightarrow} \quad \tfrac{df}{dx}\Big|_{x_L}\left(\sum_j a_j j\right) \quad = \tfrac{\partial f}{\partial x}$$

$$\sum_{j=-N}^{M} a_j j^{\alpha} = 0 \quad \overset{from}{\longrightarrow} \quad \tfrac{df^n}{dx^n}\left(\sum_j a_j j^{\alpha}\right) \quad = \tfrac{\partial f}{\partial x} + 0\cdot \tfrac{df^n}{dx^n}$$

Therefore we get a system to determine the coefficients a_j of the stencil, where $\alpha = 2$, $3, \ldots, N+M$, there are then $N+M+1$ linear algebraic equations with $N+M+1$ unknowns a_j, $j = -N, \ldots, M$. The unique solution will determine the coefficients a_j of the finite difference approximation.

$$\begin{bmatrix} \underline{j}^0 \\ \underline{j}^1 \\ \underline{j}^2 \\ \vdots \\ \underline{j}^{N+M} \end{bmatrix} \cdot \begin{bmatrix} a_{-N} \\ a_{-(N+1)} \\ a_{-(N+2)} \\ \vdots \\ a_M \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad \text{with } \underline{j} = \begin{bmatrix} -N & -N+1 & \cdots & M \end{bmatrix}$$

From the above Taylor series expansion (Eq. (2.3)), we have

$$\frac{1}{\Delta x}\sum_{j=-N}^{M} a_j f\left(x_l + j\Delta x\right) = \left(\frac{\partial f}{\partial x}\right)_l + \mathcal{O}\left(\Delta x^{N+M}\right)$$

The order of the truncation error is therefore $N+M$.
The above procedures show that from the Taylor series expansion the maximum order of accuracy which can be achieved with a $(N+M+1)$-point-stencil is $N+M$.

2.2.3 Second Derivative

Similarly, for the second order derivative $\frac{\partial^2 f}{\partial x^2}$, we have

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_l = \frac{1}{(\Delta x)^2} \sum_{j=-N}^{M} a_j f_{l+j} + \mathcal{O}\left(\Delta x^{N+M-1}\right)$$

Using the Taylor series expanxion according to the First Derivative, we have

$$\begin{aligned}\left(\frac{\partial^2 f}{\partial x^2}\right)_l =& \frac{1}{(\Delta x)^2}\left[f(x_l)\sum_{j=-N}^{M} a_j + \Delta x \left(\frac{\partial f}{\partial x}\right)_l \sum_{j=-N}^{M} a_j j\right. \\ &+ \frac{1}{2!}(\Delta x)^2 \left(\frac{\partial^2 f}{\partial x^2}\right)_l \sum_{j=-N}^{M} a_j j^3 + \ldots \\ &+ \left.\frac{1}{(N+M)!}(\Delta x)^{N+M}\left(\frac{\partial^{N+M} f}{\partial x^{N+M}}\right)_l \sum_{j=-N}^{M} a_j j^{N+M}\right] + \mathcal{O}\left(\Delta x^{N+M-1}\right)\end{aligned}$$

After Tayler Series Expanxion and comparing the left hand side with the right hand side of the equation, the coefficients a_j can be determined by

coefficients for second derivative

$$\sum_{j=-N}^{M} a_j = 0$$

$$\sum_{j=-N}^{M} a_j j = 0$$

$$\sum_{j=-N}^{M} a_j j^2 = 2!$$

$$\sum_{j=-N}^{M} a_j j^\alpha = 0$$

where $\alpha = 3, 4, \ldots, N+M$. The maximum order of accuracy for the Second Derivative which can be achieved with a $(N+M+1)$-point-stencil is $N+M-1$, except for the central difference approximation which is $N+M$.

The same procedure can be applied to evaluate higher order derivatives.

2.2.4 Finite Difference Approximation of mixed Partial Derivatives $\frac{\partial^2 f}{\partial x \partial y}$

There are two approaches

(a) Taylor Series Expansion

(b) $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

We will only discuss the approach (b).

Using $\left(\frac{\partial f}{\partial y}\right)_{i,j} = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta y} + \mathcal{O}\left(\Delta y^2\right)$, we have

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left[\frac{f_{i,j+1} - f_{i,j-1}}{2\Delta y}\right] + \mathcal{O}\left(\Delta y^2\right)$$

$$= \frac{1}{2\Delta y}\left[\left.\frac{\partial f}{\partial x}\right|_{i,j+1} - \left.\frac{\partial f}{\partial x}\right|_{i,j-1}\right] + \mathcal{O}\left(\Delta y^2\right)$$

Applying a second order central differencing for $\frac{\partial f}{\partial x}$ under the assumption of orthogonal grids

$$\left.\frac{\partial f}{\partial x}\right|_{i,j+1} = \frac{f_{i+1,j+1} - f_{i-1,j+1}}{2\Delta x} + \mathcal{O}\left(\Delta x^2\right)$$

$$\left.\frac{\partial f}{\partial x}\right|_{i,j-1} = \frac{f_{i+1,j-1} - f_{i-1,j-1}}{2\Delta x} + \mathcal{O}\left(\Delta x^2\right)$$

we have

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{f_{i+1,j+1} - f_{i-1,j+1} - f_{i+1,j-1} + f_{i-1,j-1}}{4\Delta x \Delta y} + \mathcal{O}\left(\Delta x^2, \Delta y^2\right)$$

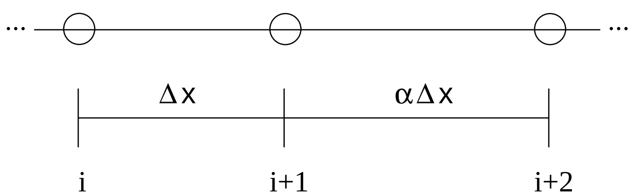
Similarly, if a first order forward differencing is used, we can then have

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{f_{i+1,j+1} - f_{i,j+1} - f_{i+1,j} + f_{i,j}}{\Delta x \Delta y} + \mathcal{O}\left(\Delta x, \Delta y\right)$$

2.2.5 Finite Differencing for Unequally Spaced Grid points

Following the procedures below, we can derive the first order approximation for the second order derivative and the second order approximation for the first order derivative.

(a) Taylor Series Expansions for $f\left(x + \Delta x\right)$ and $f\left(x + \left(1 + \alpha\right)\Delta x\right)$ around x.



(b) Sum up Taylor Series Expansions: $-\left(1 + \alpha\right) f\left(x + \Delta x\right) + f\left(x + \left(1 + \alpha\right)\Delta x\right)$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} = \frac{f_{i+2} - \left(1 + \alpha\right) f_{i+1} + \alpha f_i}{\frac{1}{2}\alpha\left(1 + \alpha\right)\left(\Delta x\right)^2} + \mathcal{O}\left(\Delta x\right)$$

(c) Replace $\left(\frac{\partial^2 f}{\partial x^2}\right)_i$ in the Taylor series expansion for $f\left(x + \Delta x\right)$ by the above equation, we have

$$\Rightarrow \quad \frac{\partial f}{\partial x} = \frac{-f_{i+2} + \left(1 + \alpha\right)^2 f_{i+1} - \alpha\left(\alpha + 2\right) f_i}{\alpha\left(1 + \alpha\right)\Delta x} + \mathcal{O}\left(\Delta x^2\right)$$

2.3 Consistency analysis of the schemes

2.3.1 Example I: FTCS

Heat conduction equation

$$\frac{\partial T}{\partial t} = \gamma \frac{\partial^2 T}{\partial x^2}$$

Scheme

FTCS: Forward in Time and Central in Space

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \gamma \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} \qquad \text{(FDE)}$$

Expand each T in a Taylor series expansion about T_i^n (i: position, n: time)

$$T_i^{n+1} = T_i^n + \left(\frac{\partial T}{\partial t}\right)_i^n \Delta t + \left(\frac{\partial^2 T}{\partial t^2}\right)_i^n \frac{(\Delta t)^2}{2!} + \mathcal{O}(\Delta t^3)$$

$$T_{i+1}^n = T_i^n + \left(\frac{\partial T}{\partial x}\right)_i^n \Delta x + \left(\frac{\partial^2 T}{\partial x^2}\right)_i^n \frac{(\Delta x)^2}{2!} + \left(\frac{\partial^3 T}{\partial x^3}\right)_i^n \frac{(\Delta x)^3}{3!} + \mathcal{O}(\Delta x^3)$$

$$T_{i-1}^n = T_i^n - \left(\frac{\partial T}{\partial x}\right)_i^n \Delta x + \left(\frac{\partial^2 T}{\partial x^2}\right)_i^n \frac{(\Delta x)^2}{2!} - \left(\frac{\partial^3 T}{\partial x^3}\right)_i^n \frac{(\Delta x)^3}{3!} + \mathcal{O}(\Delta x^3)$$

Substituting the above Taylor series expansion into the FDE

$$\frac{1}{\Delta t}\left[\underbrace{T_i^n + \left(\frac{\partial T}{\partial t}\right)_t^n \Delta t + \left(\frac{\partial^2 T}{\partial t^2}\right)_i^n \frac{\Delta t^2}{2!} + \mathcal{O}(\Delta t^3)}_{T_i^{n+1}} - T_i^n\right]$$

$$= \frac{\gamma}{(\Delta x)^2}\left[\underbrace{T_i^n + \left(\frac{\partial T}{\partial x}\right)_i^n \Delta x + \left(\frac{\partial^2 T}{\partial x^2}\right)_i^n \frac{(\Delta x)^2}{2!} + \left(\frac{\partial^3 T}{\partial x^3}\right)_i^n \frac{(\Delta x)^3}{3!} + \mathcal{O}(\Delta x^3)}_{T_{i+1}^n} - 2T_i^n\right.$$

$$\left. + \underbrace{T_i^n - \left(\frac{\partial T}{\partial x}\right)_i^n \Delta x + \left(\frac{\partial^2 T}{\partial x^2}\right)_i^n \frac{(\Delta x)^2}{2!} - \left(\frac{\partial^3 T}{\partial x^3}\right)_i^n \frac{(\Delta x)^3}{3!} + \mathcal{O}(\Delta x^3)}_{T_{i-1}^n}\right]$$

Simplify the above equation

$$\left[\frac{\partial T}{\partial t} + \frac{\partial^2 T}{\partial t^2}\frac{\Delta t}{2!} + \mathcal{O}(\Delta t)^2\right] = \gamma\left[\frac{\partial^2 T}{\partial x^2} + \mathcal{O}(\Delta x^2)\right] \text{ or}$$

$$\frac{\partial T}{\partial t} = \gamma \frac{\partial T}{\partial x^2} - \frac{\Delta t}{2}\frac{\partial^2 T}{\partial t^2} + \mathcal{O}\left[(\Delta t^2), (\Delta x^2)\right]$$

If $\begin{matrix}\Delta t \to 0 \\ \Delta x \to 0\end{matrix}$ the FDE $\to$ PDE.

We have $\frac{\partial T}{\partial t} = \gamma \frac{\partial^2 T}{\partial x^2}$, the method is therefore consistent.

2.3.2 Example II: CTCS

Heat conduction equation

$$\frac{\partial T}{\partial t} = \gamma \frac{\partial^2 T}{\partial x^2}$$

The Dufort-Frankel Scheme (Central in Time and Central in Space)

CT<u>CS</u>

↑ γ with modification

$$\frac{T_i^{n+1} - T_i^{n-1}}{2\Delta t} = \gamma \frac{T_{i+1}^n - 2\frac{T_i^{n+1} + T_i^{n-1}}{2} + T_{i-1}^n}{\Delta x^2} \qquad \text{(FDE)}$$

Expand T_i^{n+1}, T_i^{n-1}, T_{i-1}^n and T_{i+1}^n in a Taylor series about T_i^n and substitute the results into the above FDE, we have

$$\frac{\partial T}{\partial t} + \gamma \frac{\partial^2 T}{\partial t^2} \left(\frac{\Delta t}{\Delta x}\right)^2 = \gamma \frac{\partial^2 T}{\partial x^2} + \mathcal{O}\left[(\Delta t^2), (\Delta x^2)\right] \text{ or}$$

$$\frac{\partial T}{\partial t} = \gamma \frac{\partial^2 T}{\partial x^2} + \mathcal{O}\left[(\Delta t^2), (\Delta x^2), \left(\frac{\Delta t}{\Delta x}\right)^2\right]$$

The method is consistent if only Δt and Δx approach zero and if $\frac{\Delta t}{\Delta x} \to 0$.
If $\left(\frac{\Delta t}{\Delta x}\right) \to C$ (constant), we have

$$\frac{\partial T}{\partial t} + \gamma C^2 \frac{\partial^2 T}{\partial t^2} = \gamma \frac{\partial^2 T}{\partial x^2} \qquad \underline{\text{hyperbolic equation}}$$

In this case, the method is not consistent.

2.4 Von Neumann Stability Analysis

When one applies finite difference methods blindly, it is very easy to create one that is unstable. That is very small error can grow until a solution variable "blows up". Von Neumann stability analysis is one of those easy methods that give a necessary condition for stability. Although the actual stability requirement may be more restrictive than the one obtained from the Von Neumann stability analysis, the results from the Von Neumann stability analysis can provide very useful insight on stability requirements.

Von Neumann analysis is derived from a Fourier series representation of a finite difference equation. The decay or growth of the amplification factor indicates whether or not the numerical algorithm is stable. In general, the linearization of the equation is a requirement for the application of the von Neumann analysis. A locally linearized equation should be used for a nonlinear equation. The procedures are quite simple.

The followings are three examples of using von Neumann stability analysis to decide the stability of a finite difference method.

2.4.1 Example I: The unsteady heat conduction equation in one-dimension

$$\frac{\partial U}{\partial t} = \gamma \frac{\partial^2 U}{\partial x^2}$$

The equation can be discretized in FTCS formulation as

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = \gamma \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{(\Delta x)^2} \tag{2.5}$$

Assuming a Fourier component for U_i^n in space where I is the imaginary number

$$\begin{aligned} U_i^n &= U^n e^{I\theta i} \\ U_i^{n+1} &= U^{n+1} e^{I\theta i} \\ U_{i\pm 1}^n &= U^n e^{I\theta(i\pm 1)} \end{aligned}$$

By substituting the Fourier components into the above equation (2.5), we have

$$\begin{aligned} U^{n+1} &= U^n \left[1 + 2d(\cos\theta - 1)\right] \\ \rightarrow \ U^{n+1} &= GU^n \end{aligned}$$

where

$$d = \frac{\Delta t}{(\Delta x)^2}\gamma, \quad \cos\theta = \frac{e^{I\theta} + e^{-I\theta}}{2}, \quad G = 1 - 2d\left(1 - \cos\theta\right)$$

The stability requirement is that the value of the amplification factor G, must be bounded for all values of θ. That is

$$|G| \leq 1$$

or

$$1 - 2d\left(1 - \cos\theta\right) \leq 1$$

and

$$1 - 2d\left(1 - \cos\theta\right) \geq -1$$

for all θ

$$\Downarrow$$

Therefore the stability condition for the FTCS scheme is $d \leq \frac{1}{2}$ or $\gamma\Delta t \leq \frac{1}{2}(\Delta x)^2$.

When there are more than two time levels involved in a FDE, after applying the von Neumann stability analysis the equations are expressed in a matrix form such as (for a three time level discretization $U^{n+1} = AU^n + BU^{n-1}$)

$$\begin{bmatrix} U^{n+1} \\ U^n \end{bmatrix} = \begin{bmatrix} A & B \\ 1 & 0 \end{bmatrix} \begin{bmatrix} U^n \\ U^{n-1} \end{bmatrix}$$

Now the amplification factor G is a matrix

$$G = \begin{bmatrix} A & B \\ 1 & 0 \end{bmatrix}$$

The stability criterion is then that the eigenvalues of $G\left(\lambda_i\right)$ must satisfy the stability condition $|\lambda_i| \leq 1$.

2.4.2 Example II: The 1-D "wave equation"

$$\frac{\partial U}{\partial t} + a\frac{\partial U}{\partial x} = 0 \quad \text{with } a > 0$$

Several methods are considered for the von Neumann stability analysis in the following.

- Euler Explicit Method (FTFS: Forward in Time and Forward in Space)

FDE $\Rightarrow$ $\quad \frac{U_i^{n+1}-U_i^n}{\Delta t} + a\frac{U_{i+1}^n - U_i^n}{\Delta x} = 0 \quad a > 0$

with $\quad U_i^n = U^n e^{I\theta i}, \quad U_i^{n+1} = U^{n+1}e^{I\theta i}, \quad U_{i+1}^n = U^n e^{I\theta(i+1)}$

we have

$$0 = \frac{U^{n+1}e^{I\theta i} - U^n e^{I\theta i}}{\Delta t} + a\frac{U^n e^{I\theta(i+1)} - U^n e^{I\theta i}}{\Delta x}$$

$$0 = \frac{U^{n+1} - U^n}{\Delta t} + \frac{a}{\Delta x}\left[U^n e^{I\theta} - U^n\right]$$

$$U^{n+1} - U^n + \frac{a\Delta t}{\Delta x}\left[U^n e^{I\theta} - U^n\right] = 0$$

$$U^{n+1} = U^n\left[1 + \frac{a\Delta t}{\Delta x}\left(-e^{I\theta} + 1\right)\right] = GU^n$$

where

$$G = 1 + \frac{a\Delta t}{\Delta x} - \frac{a\Delta t}{\Delta x}e^{I\theta} = 1 + d(1 - \cos\theta) - id\sin\theta$$

Since we have $|G| > 1$, FTFS method for this problem is **unconditional unstable**!

- The first upwind differencing method (FTBS: Foward in Time, Backward in Space)

FDE $\Rightarrow$ $\quad \frac{U_i^{n+1}-U_i^n}{\Delta t} = -a\frac{U_i^n - U_{i-1}^n}{\Delta x}$

Application of the von Neumann stability analysis yields

$$U^{n+1} = U^n\left(1 - c + ce^{-I\theta}\right)$$

where $c = \frac{a\Delta t}{\Delta x}$, the Courant number. The stability condition is therefore $c \leq 1$. Please note that for a case of $a < 0$, a forward differencing in space must be used.

- The Lax method

Using the von Neumann stability analysis, we can show that Euler's FTCS method is unconditional unstable, i.e.

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = -a\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x} \qquad \text{unconditional unstable!}$$

but if we replace U_i^n by $U_i^n = \frac{1}{2}\left(U_{i+1}^n + U_{i-1}^n\right)$, we have

$$U_i^{n+1} = \frac{1}{2}\left(U_{i+1}^n + U_{i-1}^n\right) - \frac{a\Delta t}{2\Delta x}\left(U_{i+1}^n - U_{i-1}^n\right)$$

Von Neumann stability analysis shows that the Lax method is conditional stable when $c \leq 0$.

- The Lax-Wentroff method

  From Taylor series expansion, we have

$$U(x, t + \Delta t) = U(x, t) + \frac{\partial U}{\partial t}\Delta t + \frac{\partial^2 U}{\partial t^2}\frac{(\Delta t)^2}{2!} + \mathcal{O}(\Delta t)^3$$

  i.e.

$$U_i^{n+1} = U_i^n + \frac{\partial U}{\partial t}\Delta t + \frac{\partial^2 U}{\partial t^2}\frac{(\Delta t)^2}{2!} + \mathcal{O}(\Delta t)^3$$

  From the "wave equation", we have

$$\frac{\partial U}{\partial t} = -a\frac{\partial U}{\partial x}$$

$$\rightarrow \frac{\partial^2 U}{\partial t^2} = -a\frac{\partial}{\partial t}\left(\frac{\partial U}{\partial x}\right) = -a\frac{\partial}{\partial x}\left(\frac{\partial U}{\partial t}\right) = a^2\frac{\partial^2 U}{\partial x^2}$$

  Substituting $\frac{\partial U}{\partial t}$ and $\frac{\partial^2 U}{\partial t^2}$ from the above equations into the Taylor series expansion

$$U_i^{n+1} = U_i^n + \left(-a\frac{\partial U}{\partial x}\right)\Delta t + \frac{(\Delta t)^2}{2}\left(a^2\frac{\partial^2 U}{\partial x^2}\right)$$

  Using 2^{nd}–order central differencing method for the spatial derivatives, we have

$$U_i^{n+1} = U_i^n - a\Delta t\left[\frac{U_{i+1}^n - U_{i-1}^n}{2\Delta x}\right] + \frac{1}{2}a^2(\Delta t)^2\left[\frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{(\Delta x)^2}\right]$$

  This explicit scheme is stable when $c \leq 1$. The order of accuracy is $\mathcal{O}\left[(\Delta t)^2, (\Delta x)^2\right]$.

- Euler's implicit method (BTCS: Backward in Time, Central in Space)

  FDE $\Rightarrow$ $\frac{U_i^{n+1} - U_i^n}{\Delta t} = -\frac{a}{2\Delta x}\left(U_{i+1}^{n+1} - U_{i-1}^{n+1}\right)$

  After applying the above equation to all the grid points at the time level $n + 1$ (unknown), a set of linear algebraic equations will need to be solved. The equations can be represented in a matrix form, where the coefficient matrix is tridiagonal. The method is unconditional stable and the accuracy is of order $(\Delta t), (\Delta x)^2$

- The McCormack Method

  The method is a multi-level method and is widely used for solving fluid flow equations.

  Predictor: $\frac{U_i^* - U_i^n}{\Delta t} = -a\frac{U_{i+1}^n - U_i^n}{\Delta x}$

  Corrector: $\frac{U_i^{n+1} - U_i^{n+\frac{1}{2}}}{\frac{1}{2}\Delta t} = -a\frac{U_i^* - U_{i-1}^*}{\Delta x}$

  After the term $U_i^{n+\frac{1}{2}}$ is replaced by an average value $U_i^{n+\frac{1}{2}} = \frac{1}{2}(U_i^n + U_i^*)$ we have

  Corrector: $U_i^{n+1} = \frac{1}{2}\left[(U_i^n + U_i^*) - \frac{a\Delta t}{\Delta x}\left(U_i^* - U_{i-1}^*\right)\right]$

  The method is explicit and conditional stable when $a\frac{\Delta t}{\Delta x} \leq 1$

- Splitting methods and Multi-step Methods

  Splitting methods: It is to split a finite difference scheme into a sequence of one-dimensional operations and then to solve them sequentially.

  e.g. ADI method (Alternative Direction Implicit method) for $\frac{\partial U}{\partial t} = \alpha \left[\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right]$

$$\frac{U_{i,j}^{n+\frac{1}{2}} - U_{i,j}^{n}}{\frac{\Delta t}{2}} = a \left[\frac{U_{i+1,j}^{n+\frac{1}{2}} - 2U_{i,j}^{n+\frac{1}{2}} + U_{i-1,j}^{n+\frac{1}{2}}}{(\Delta x)^2} + \frac{U_{i,j+1}^{n} - 2U_{i,j}^{n} + U_{i,j-1}^{n}}{(\Delta y)^2} \right]$$

  and

$$\frac{U_{i,j}^{n+1} - U_{i,j}^{n+\frac{1}{2}}}{\frac{\Delta t}{2}} = a \left[\frac{U_{i+1,j}^{n+\frac{1}{2}} - 2U_{i,j}^{n+\frac{1}{2}} + U_{i-1,j}^{n+\frac{1}{2}}}{(\Delta x)^2} + \frac{U_{i,j+1}^{n+1} - 2U_{i,j}^{n+1} + U_{i,j-1}^{n+1}}{(\Delta y)^2} \right]$$

  Multi-step Methods: These methods have in general two steps. In the first step, a temporary value for the dependent variable is predicted. In the second step, a corrected value is computed to give the final value of the dependent variable.
  e.g. The McCormack Method.

2.5 Fourier Error Analysis

In order to choose a finite difference scheme for a given application, we must be able to access the accuracy of the scheme. Although a leading error term can be determined from a Taylor series expansion, this measure only provides very limited information. The Fourier error analysis can describe the error behavior of a finite difference scheme.

An arbitrary periodic function can be decomposed into its Fourier components, which are in the form $e^{i\alpha x}$, where α is the wavenumber and i the imaginary number.

Let us consider the following function and its first derivative

$$f(x) = e^{i\alpha x}$$

$$\frac{df}{dx} = i\alpha e^{i\alpha x} \qquad \text{(analytical solution of the first derivative)}$$

In the following, the numerical solution of the first derivative is determined and compared with the analytical.

2.5.1 Using a second-order central differencing

$$\left(\frac{df}{dx}\right)_j = \frac{f_{j+1} - f_{j-1}}{2\Delta x} + \mathcal{O}\left(\Delta x^2\right)$$

$$= \frac{e^{i\alpha(j\Delta x + \Delta x)} - e^{i\alpha(j\Delta x - \Delta x)}}{2\Delta x} + \mathcal{O}\left(\Delta x^2\right)$$

$$= \frac{e^{i\alpha x_j}\left(e^{i\alpha\Delta x} - e^{-i\alpha\Delta x}\right)}{2\Delta x} + \mathcal{O}\left(\Delta x^2\right)$$

$$\left(\frac{\partial f}{\partial x}\right)_j \approx \frac{1}{2\Delta x}\left[(\cos\alpha\Delta x + i\sin\alpha\Delta x) - (\cos\alpha\Delta x - i\sin\alpha\Delta x)\right] e^{i\alpha x_j}$$

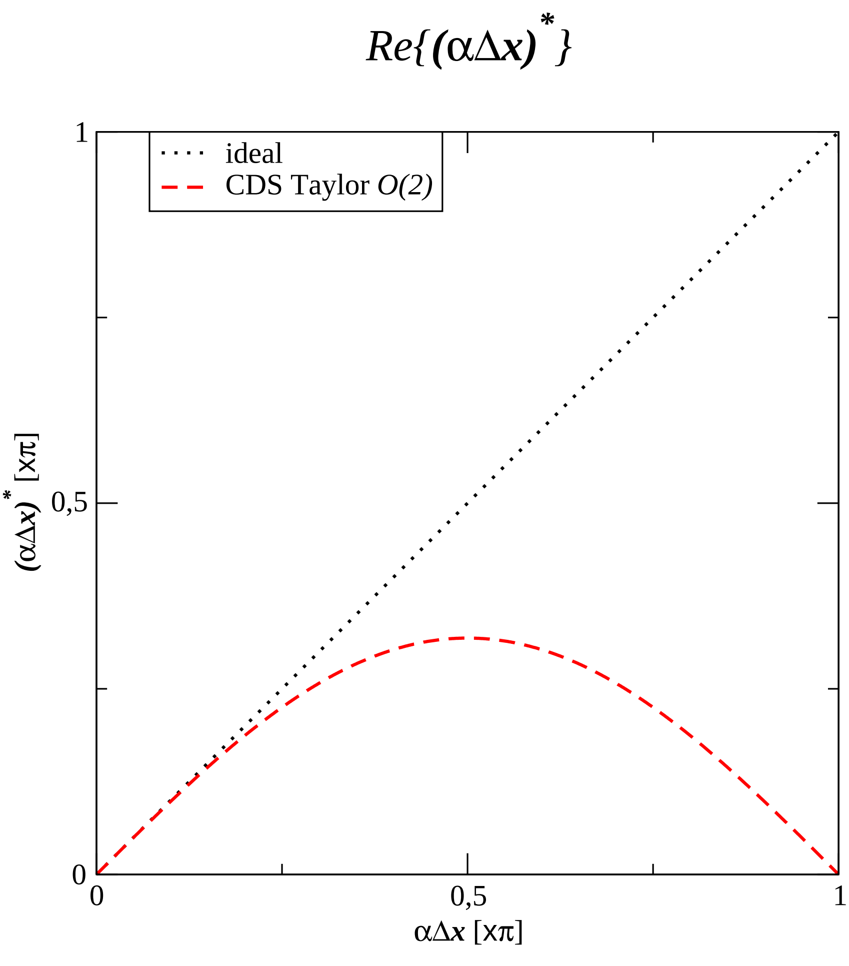
$$= i\underbrace{\frac{\sin\alpha\Delta x}{\Delta x}}_{=\alpha^*} e^{i\alpha x_j}$$

$$= i\alpha^* e^{i\alpha x_j} \qquad \text{(numerical solution of the first derivative)}$$

where $\alpha^* = \frac{\sin\alpha\Delta x}{\Delta x}$ is the effective or modified wavenumber.

Note that α^* approximates α to second-order accuracy as expected.

$$\alpha^* = \alpha - \frac{\alpha^3\Delta x^2}{6} + \dots$$



2.5.2 Using a fourth-order central differencing

$$\left.\frac{df}{dx}\right|_j = \frac{-f_{j+2} + 8f_{j+1} - 8f_{j-1} + f_{j-2}}{12\Delta x} + \mathcal{O}\left(\Delta x^4\right)$$

$$= \frac{-e^{i\alpha(x_j+2\Delta x)} + 8e^{i\alpha(x_j+\Delta x)} - 8e^{i\alpha(x_j-\Delta x)} + e^{i\alpha\left(x_j-2\Delta x\right)}}{12\Delta x} + \mathcal{O}\left(\Delta x^4\right)$$

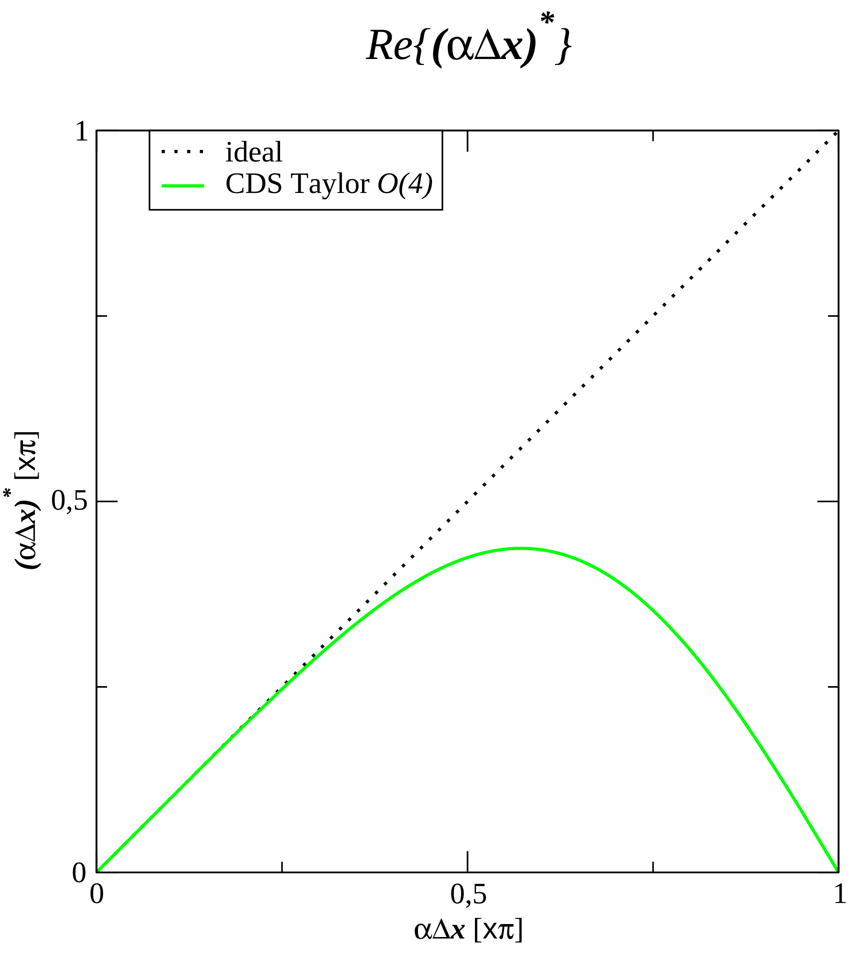
$$= \frac{e^{i\alpha x_j}\left[-e^{2i\alpha\Delta x} + 8e^{i\alpha\Delta x} - 8e^{-i\alpha\Delta x} + e^{-i\alpha 2\Delta x}\right]}{12\Delta x} + \mathcal{O}\left(\Delta x^4\right)$$

$$\left(\frac{\partial f}{\partial x}\right)_j \approx \left[-\frac{\sin\left(2\alpha\Delta x\right)}{6\Delta x} + \frac{4\sin\alpha\Delta x}{3\Delta x}\right] ie^{i\alpha x_j}$$

$$= i\alpha^* e^{i\alpha x_j}$$

$$\text{where } \alpha^* = -\frac{\sin\left(2\alpha\Delta x\right)}{6\Delta x} + \frac{4\sin\alpha\Delta x}{3\Delta x}$$

$$= \alpha - \frac{4\alpha^5\Delta x^4}{3\Delta x 5!} + \ldots$$



Note that this time α^* approximates α to fourth-order accuracy as expected.
From the above two examples, we also have noticed that the modified wavenumber is purely real when a central differencing is applied.

2.5.3 Using a second-order backward differencing

$$\left.\frac{df}{dx}\right|_j = \frac{3f_j - 4f_{j-1} + f_{j-2}}{2\Delta x} + \mathcal{O}\left(\Delta x^2\right)$$

$$= \frac{3e^{i\alpha(x_j)} - 4e^{i\alpha(x_j-\Delta x)} + e^{i\alpha(x_j-2\Delta x)}}{2\Delta x} + \mathcal{O}\left(\Delta x^2\right)$$

$$= \frac{e^{i\alpha x_j}\left[3 - 4e^{-i\alpha\Delta x} + e^{-2i\alpha\Delta x}\right]}{2\Delta x} + \mathcal{O}\left(\Delta x^2\right)$$

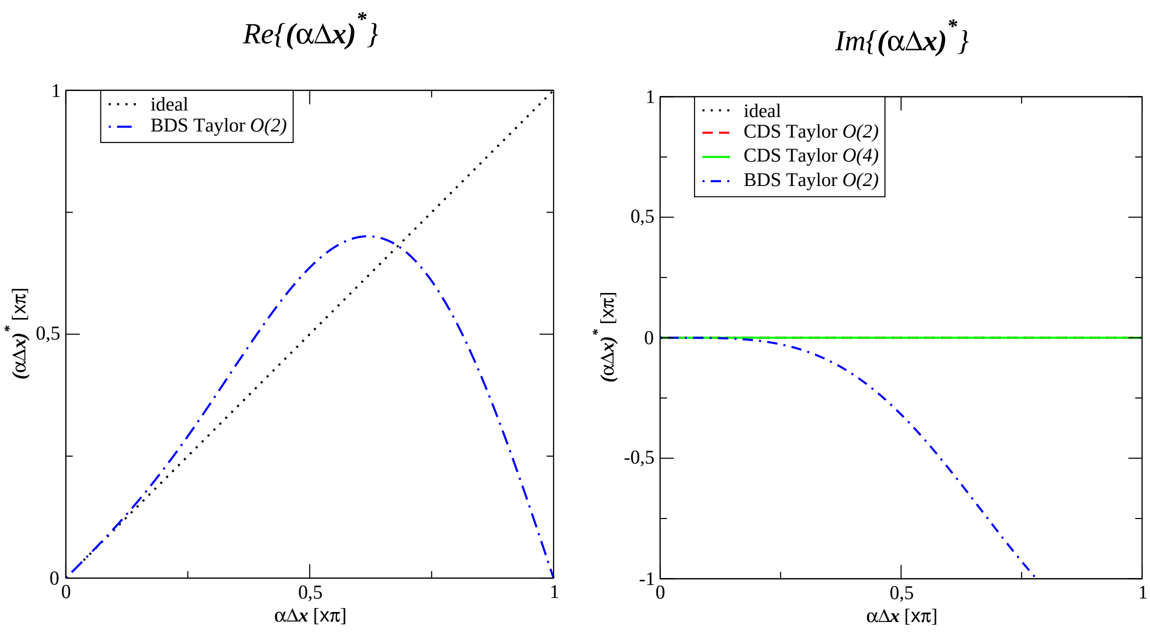
$$= \frac{e^{i\alpha x_j}\left[(3 - 4\cos\alpha\Delta x + \cos 2\alpha\Delta x) + i\left(4\sin\alpha\Delta x - \sin 2\alpha\Delta x\right)\right]}{2\Delta x} + \mathcal{O}\left(\Delta x^2\right)$$

$$\left(\frac{\partial f}{\partial x}\right)_j \approx \left[-i\frac{3 - 4\cos\alpha\Delta x + \cos 2\alpha\Delta x}{2\Delta x} + \frac{4\sin\alpha\Delta x - \sin 2\alpha\Delta x}{2\Delta x}\right] ie^{i\alpha x_j}$$

$$= i\alpha^* e^{i\alpha x_j}$$

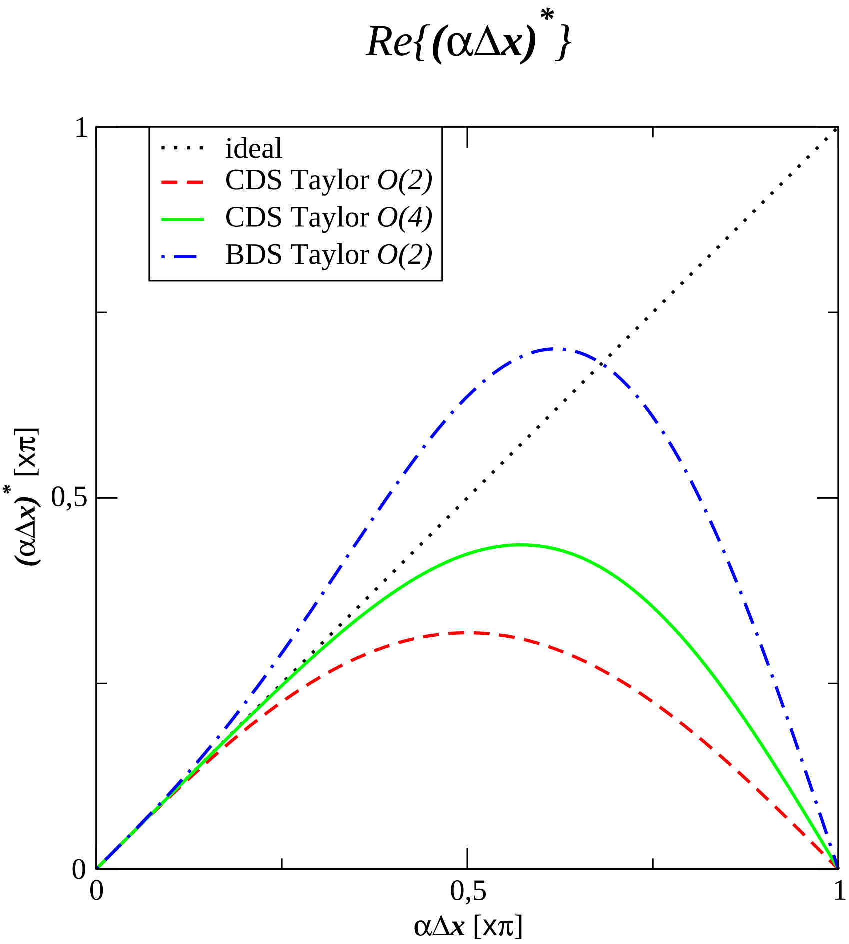
$$\text{where } \alpha^* = -\left[\frac{3 - 4\cos\alpha\Delta x + \cos 2\alpha\Delta x}{2\Delta x}\right] i + \frac{4\sin\alpha\Delta x - \sin 2\alpha\Delta x}{2\Delta x}$$

$$= -i\frac{17}{4!}\alpha^4\Delta x^3 + \alpha + \frac{\alpha^3}{3}\Delta x^2$$



When a non-central differencing is applied, the modified wavenumber is complex, with the imaginary component being entirely error. The imaginary component can be avoided by using central differencing schemes. In general, the modified wavenumber has both real and imaginary parts. By assuming an harmonic function $e^{i\alpha^*\Delta x}$, the numerical error in the phase (dispersion) is determined from the real part of the modified wavenumber and the numerical error in amplitude (dissipation) is determined from the imaginary part of the modified wavenumber.

All differencing methods are shown here:



For small α^*, the approximation of α by α^* is good. For higher α the accuracy of α^* decreases. For high-order schemes, however, the accuracy of α^* prolongs for higher wavenumbers. That's why greater wavelenghts or lower frequencies can be computed more accurate by high order schemes.

One can see from the image above, how many points per wavelength (PPW) is needed to get an accurate approximation for a distinct frequency. PPW is determined by $\frac{\lambda}{\Delta x}$, where λ is the wavelength. The definition of the wavenumber is given by

$$\alpha = \frac{2\pi}{\lambda} \Rightarrow \lambda = \frac{2\pi}{\alpha} \qquad \Rightarrow \qquad \text{PPW} = \frac{2\pi}{\alpha \Delta x}$$

hence for $\alpha \Delta x = 0.5\pi$ 4 PPW are necessary.

Considering the image above, for a second-order central differencing scheme the highest resolvable wavenumber is about $\alpha \Delta x \approx \frac{\pi}{10}$, that are about 20 PPW. For a fourth-order central differencing scheme the highest resolvable wavenumber is about $\alpha \Delta x \approx \frac{\pi}{6}$, that are about 12 PPW.

3 Optimized Spatial Discretization

Literature: Tam [2], Lele [1]

Consider the model wave equation.

$$\frac{\partial U}{\partial t} + c\frac{\partial U}{\partial x} = 0 \tag{3.1}$$

with the fundamental solution

$$U(x,t) = e^{i(\alpha x - \omega t)} \tag{3.2}$$

Without the loss of generality, we assume that $c > 0$ with. Substituting (3.2) into (3.1) we have

$$e^{i(\alpha x - \omega t)}(-i\omega + ic\alpha) = 0 \tag{3.3}$$

To let this equation be true and therewith equation (3.1), it's required that

$$\omega = c\alpha \ .$$

This is the ideal dispersion relation for the model wave equation.

After using the spatial discretization for the first order spatial derivative on a uniform grid of spacing Δx we get

$$\left(\frac{\partial U}{\partial x}\right)_l \approx \frac{1}{\Delta x}\sum_{j=-N}^{M} a_j U_{l+j} \tag{3.4}$$

where $U_{l+j} = U((l+j)\Delta x, t)$, $\left(\frac{\partial U}{\partial x}\right)_l = \frac{\partial U(l\Delta x,t)}{\partial x}$ and a_j are coefficients which will be determined according to required order of accuracy and other properties.
Substituting the above equation (3.4) into equation (3.1), we obtain a system of semi-discrete ordinary differential equations

$$\left(\frac{dU}{dt}\right)_l + \frac{c}{\Delta x}\sum_{j=-N}^{M} a_j U_{l+j} = 0 \tag{3.5}$$

Consider the fundamental solution of equation (3.5) with a given wave number α in the form of

$$U_s(l\Delta x, t) = e^{i(\alpha l\Delta x - \omega t)} \tag{3.6}$$

where s stands for semi-discrete to distinguish it from the fundamental solution given in equation (3.2). Substituting equation (3.6) into equation (3.5), we have

$$e^{i(\alpha l\Delta x - \omega t)}\left[-i\omega + \frac{c}{\Delta x}\sum_{j=-N}^{M}\alpha_j e^{i\alpha j\Delta x}\right] = 0$$

In comparison with the solution of the original solution (3.3), we see that the effective numerical wavenumber $\bar{\alpha}$ is defined as

$$\begin{aligned} -i\omega + ic\bar{\alpha} &= -i\omega + \frac{c}{\Delta x}\sum_{j=-N}^{M}\alpha_j e^{i\alpha j\Delta x} \\ \Rightarrow \bar{\alpha} &= -\frac{i}{\Delta x}\sum_{j=-N}^{M}\alpha_j e^{i\alpha j\Delta x} \end{aligned} \tag{3.7}$$

Then the dispersion relation of the semi discrete equation (3.5) is given as

$$\omega = c\overline{\alpha}$$

From the above dispersion relation, we can write the fundamental solution with a given wavenumber α of the semi-discrete equation (equation (3.5)) as

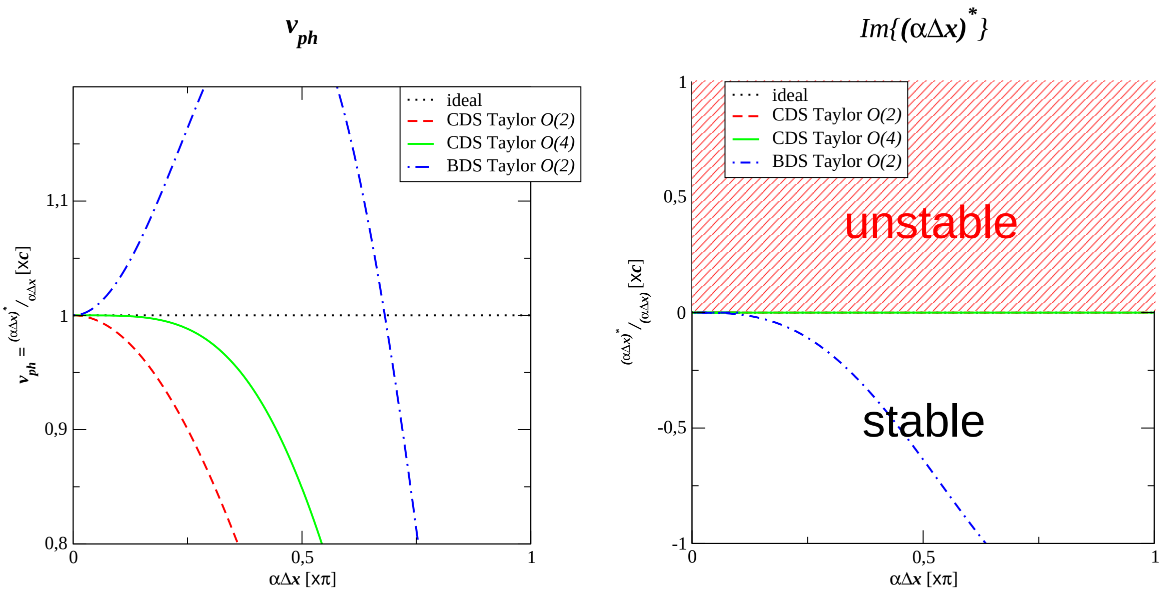
$$U_s\left(l\Delta x, t\right) = e^{i(\alpha l\Delta x - c\overline{\alpha}t)} = e^{i[\alpha l\Delta x - c(\bar{\alpha}_r + i\bar{\alpha}_i)t]} = e^{i\alpha\left[l\Delta x - c\left(\frac{\bar{\alpha}_r}{\alpha}\right)t\right]} e^{c\bar{\alpha}_i t} \tag{3.8}$$

where $\overline{\alpha} = \overline{\alpha}_r + i\overline{\alpha}_i$.

The phase velocity $v_p = c\left(\overline{\alpha}_r/\alpha\right)$ is no longer a constant, unless that $\bar{\alpha}_r = \alpha$ (not possible in a numerical solution). Waves with different wavenumbers therefore propagate with different speeds, above or below the speed of sound. For an initial wave with different wavenumbers, its wave shape is no longer possible to be kept unchanged. That's why an pulse isn't kept unmodified. This phenomenon is called dispersion, whereupon the real part of the numerical wavenumber $\bar{\alpha}_r$ is responsible for. [2]

Note that if $\overline{\alpha}_i \neq 0$, in addition to traveling with different speeds, waves with different wavenumbers are now growing or diminishing with different factors. This phenomenon is called dissipation, at which the imaginary part of the numerical wavenumber $\bar{\alpha}_i$ is responsible for.

$c\overline{\alpha}_i > 0$	unstable (growing)
$c\overline{\alpha}_i < 0$	stable (diminishing)



Comparing fundamental solutions from semi-discrete equation (3.8) and analytical equation (3.2), we have

$$\frac{U_s\left(l\Delta x, t\right)}{U\left(l\Delta x, t\right)} = e^{ic(\alpha - \overline{\alpha}_r)t} e^{c\overline{\alpha}_i t} \tag{3.9}$$

Therefore the phase and the magnitude errors introduced by the spatial discretization of equation (3.1) are

$$\left|\Phi_{s(t)} - \Phi\right| = \left|c\left(\alpha - \overline{\alpha}_r\right)t\right| \qquad \text{(phase)}$$
$$G\left(t\right) = e^{c\overline{\alpha}_i t} \qquad \text{(amplitude)}$$

In order to minimize the dispersion and the dissipation errors in a numerical scheme, we would like to revisit equation (3.7)

$$\overline{\alpha} = \frac{-i}{\Delta x} \sum_{j=-N}^{M} a_j e^{i\alpha j \Delta x}$$

where $\overline{\alpha}$: the effective wave number of the finite difference approximation.
$\overline{\alpha}\Delta x$: is a periodic function with a period of 2π.

Equation (3.7) can be rewritten as

$$\begin{aligned}
\overline{\alpha}\Delta x &= -i \sum_{j=-N}^{M} a_j e^{i\alpha j\Delta x} \\
&= -i \sum_{j=-N}^{M} \left[a_j \cos\left(\alpha j \Delta x\right) + i a_j \sin\left(\alpha j \Delta x\right)\right] \\
&= \sum_{j=-N}^{M} a_j \sin\left(\alpha j \Delta x\right) - i \sum_{j=-N}^{M} a_j \cos\left(\alpha j \Delta x\right)
\end{aligned}$$

By considering $\overline{\alpha} = \overline{\alpha}_r + i\overline{\alpha}_i$ and comparing the left hand side of the equation with the right hand side we obtain:

$$\boxed{\overline{\alpha}_r \Delta x = \sum_{j=-N}^{M} a_j \sin\left(\alpha j \Delta x\right)} \tag{3.10}$$

$$\boxed{\overline{\alpha}_i \Delta x = -\sum_{j=-N}^{M} a_j \cos\left(\alpha j \Delta x\right)} \tag{3.11}$$

So one aim for optimization is to let $\alpha_r \Delta x \to \alpha \Delta x$ (minimization of dispersion error) and $\alpha_i \Delta x \to 0$ (minimization of the dissipation error).

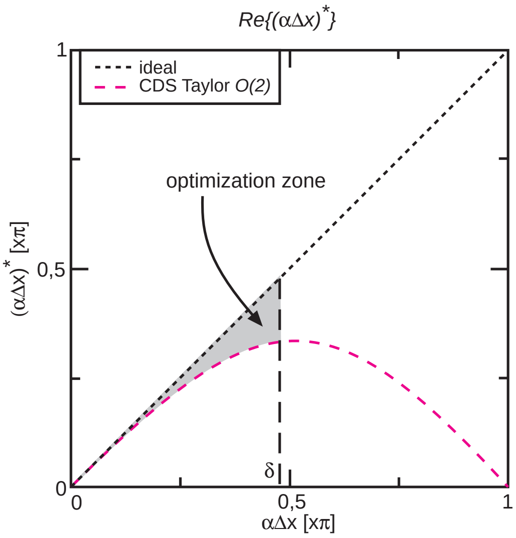
For optimization purpose, a range of wavenumbers in which the integrated error should be small has to be defined. In the following picture the optimization range is set to δ in general.
To assure minimum local truncation errors over a given range of wavenumbers, e.g. waves with wavelength longer than $4\Delta x$ (i.e. $\lambda \geq 4\Delta x$) or $\alpha\Delta x \leq \frac{\pi}{2} = \delta$, the **integrated error** is defined with the Euclidian Norm as

$$\begin{aligned}
E := &\int_{-\delta}^{\delta} \left|\alpha_r \Delta x - \overline{\alpha}_r \Delta x\right|^2 d\left(\alpha \Delta x\right) \\
&+ \int_{-\delta}^{\delta} \left|\overline{\alpha}_r \Delta x + sgn\left(c\right) \exp\left[-\ln 2 \left(\frac{\alpha\Delta x - \pi}{\sigma}\right)^2\right]\right|^2 d\left(\alpha\Delta x\right)
\end{aligned} \tag{3.12}$$

in which the second summand represents the imaginary part of the numerical wavenumber $\bar{\alpha}_i \Delta x$.

For now, let's consider a **central finite difference scheme** ($\overline{\alpha}_i = 0$, $M = N$).

$$\boxed{E := \int_{-\delta}^{\delta} \left|\alpha\Delta x - \overline{\alpha}\Delta x\right|^2 d\left(\alpha\Delta x\right)} \tag{3.13}$$



In order to have $\overline{\alpha}_i = 0$, the coefficients a_j must be antisymmetric, i.e.,

$$\begin{aligned}
a_0 &= 0 \\
a_{-j} &= -a_j \\
\overline{\alpha}\Delta x = \overline{\alpha}_r \Delta x &= \sum_{j=-N}^{N} a_j \sin(\alpha j \Delta x) \\
&= 2\sum_{j=1}^{N} a_j \sin(\alpha j \Delta x)
\end{aligned}$$

Substitute the above expression for $\overline{\alpha}\Delta x$ into equation (3.13), we have

$$E = \int_{-\delta}^{\delta} \left[\alpha\Delta x - 2\sum_{j=1}^{N} a_j \sin(\alpha j \Delta x) \right]^2 d(\alpha\Delta x) \tag{3.14}$$

The conditions for E to be minimum are

$$\frac{\partial E}{\partial a_j} = 0, \qquad j = 1, 2 \ldots N.$$

Theoretically there are N equations for N coefficients a_j. If a_j are determined solely by Taylor Series expansion, the order of accuracy that can be achieved by a seven-point-stencil is 6. By considering the coefficients' antisymmetry $a_{-j} = -a_j$ as well as $a_0 = 0$, the coefficients for the first derivative can be evaluated with the box on page 14

$$\left.\begin{aligned}
2\sum_{j=1}^{3} a_j j &= 1 \\
2\sum_{j=1}^{3} a_j j^3 &= 0 \\
2\sum_{j=1}^{3} a_j j^5 &= 0
\end{aligned}\right\} \qquad
\begin{aligned}
a_1 &= \tfrac{45}{60} \\
a_2 &= -\tfrac{9}{60} \\
a_3 &= \tfrac{1}{60}
\end{aligned}$$

$$\left(\frac{\partial f}{\partial x}\right)_l = \frac{f_{l+3} - 9f_{l+2} + 45f_{l+1} - 45f_{l-1} + 9f_{l-2} - f_{l-3}}{60\Delta x} + \mathcal{O}\left(\Delta x^6\right)$$

If we combine the Taylor series method and the wavenumber space optimization method for a seven-point-stencil scheme,

1) keep the order of accuracy 4 and solve the resulting linear system of equations (choose a_1 as a free parameter (without loss of generality)):

$$\left.\begin{aligned} 2\sum_{j=1}^{3} a_j j &= 1 \\ 2\sum_{j=1}^{3} a_j j^3 &= 0 \end{aligned}\right\} \quad \mathcal{O}\left(\Delta x^4\right)$$

$$\begin{aligned} \Rightarrow a_2 &= \quad \tfrac{9}{20} - \tfrac{4}{5}a_1 \\ a_3 &= -\tfrac{2}{15} + \tfrac{1}{5}a_1 \end{aligned}$$

2) substitute a_2 and a_3 into equation (3.14), the integrated error E is a function of a_1 along. In the following the range of optimization is set to $\delta = \frac{\pi}{2}$. The value of a_j can be decided by the condition

$$\boxed{\frac{\partial E}{\partial a_1} = 0} \tag{3.15}$$

i.e.

$$E = 2\int_0^{\delta} \left[\beta - 2\sum_{j=1}^{3} a_j \sin(j\beta)\right]^2 d\beta \qquad \text{where } \beta = \alpha\Delta x$$

$$\xrightarrow{\frac{\partial E}{\partial a_1}=0} \quad -4\int_0^{\delta} \left[\beta - 2\sum_{j=1}^{3} a_j \sin(j\beta)\right] \left[\sum_{j=1}^{3} \frac{\partial a_j}{\partial a_1} \sin(j\beta)\right] d\beta = 0 \tag{3.16}$$

and thus

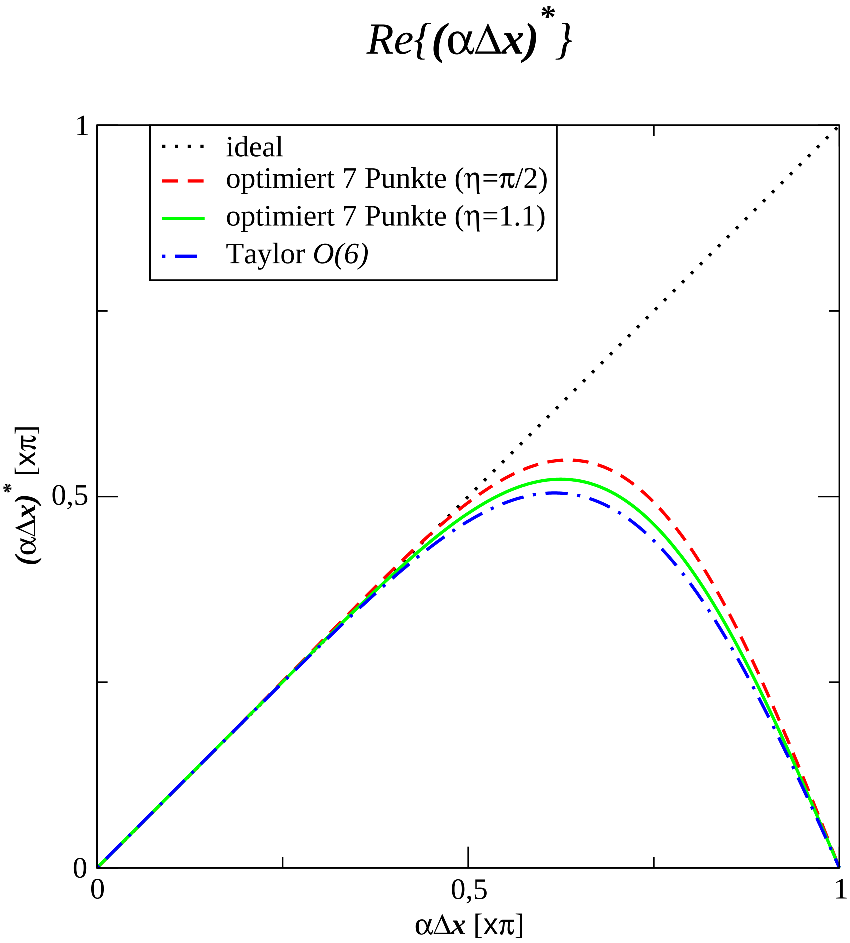
$$\boxed{\begin{aligned} &a_0 = 0 \\ &a_1 = -a_{-1} = 0.79926643 \\ &a_2 = -a_{-2} = -0.18941314 \\ &a_3 = -a_{-3} = 0.02651995 \end{aligned}}$$

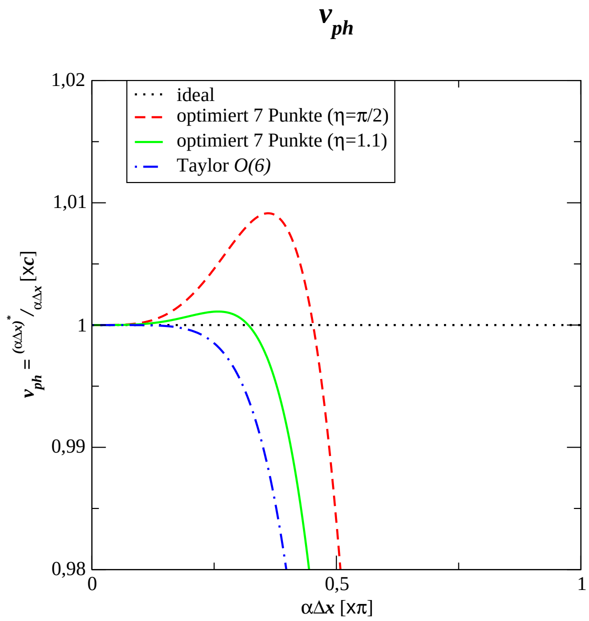
The numerical scheme developed above by combining the Taylor series method with the wavenumber space optimization method is called the **Dispersion Relation Preserving scheme (DRP)** [2], which minimizes the dispersion error.

References

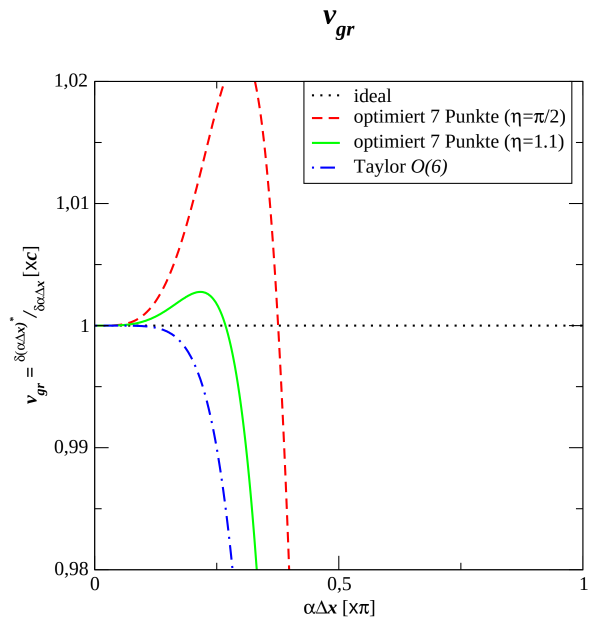
[1] S.K. Lele. Compact Finite Difference Schemes with Spectral-like Resulution. *Journal of Computational Physics*, 103:16–42, 1992.

[2] C. K. W. Tam, C. Webb, and T. Z. Dong. A Study of Short Wave Components in Computational Aeroacoustics. *Journal of Computational Acoustics*, 1:1–30, März 1993.

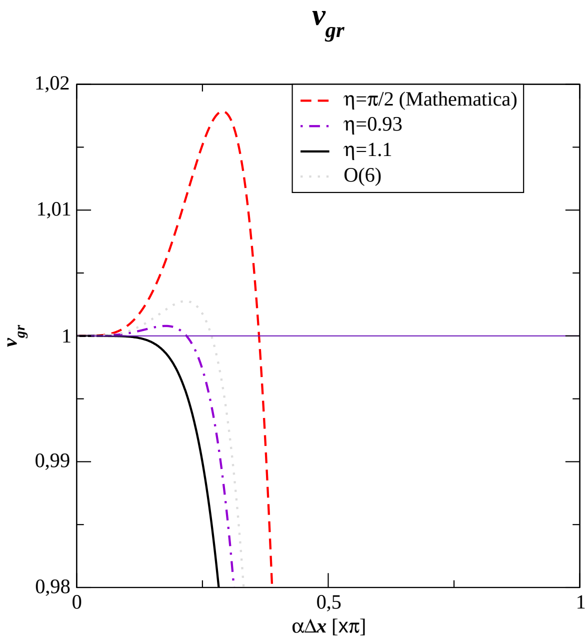




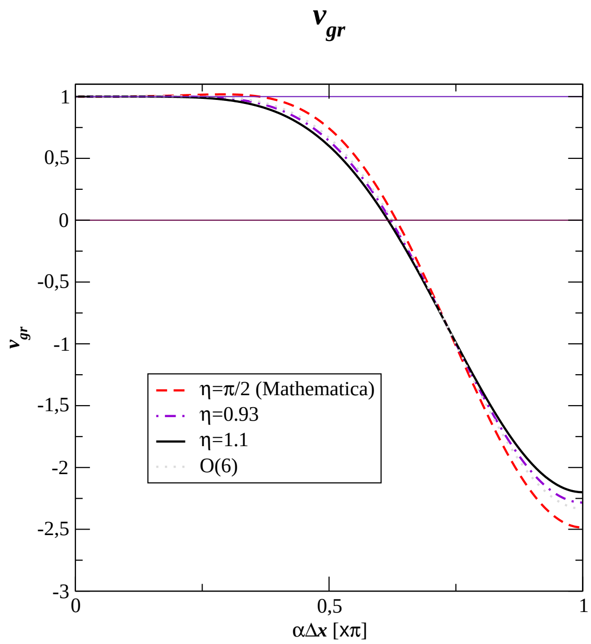
phase velocity



group velocity



group velocity



group velocity

4 Optimized Time Discretization

Literature: Tam [4], Hu [1, 2]

There are two types of explicit time-marching schemes.

(a) Single-step scheme (e.g. Runge-Kutta method)

(b) Multi-step scheme (e.g. Adams-Bashford method)

We will discuss the optimized multi-step method.

Suppose $\vec{u}(t)$ is an unknown vector and the time axis is divided into a uniform grid time step Δt. Assuming that $\vec{u}$ and $\frac{d\vec{u}}{dt}$ are known at time level n, $n-1$, $n-2$ and $n-3$. One can get the value of the next time step $(n+1)$ by

$$\vec{u}^{(n+1)} = \vec{u}^{(n)} + \underline{\Delta t \sum_{j=0}^{3} b_j \left(\frac{d\vec{u}}{dt}\right)^{(n-j)}} \tag{4.1}$$

↑
a weighted average of the time derivatives
(4-level finite difference approximation)

Four coefficients, b_0, b_1, b_2 and b_3 need to be determined. Using the Taylor series expansion around n for the derivative of time, the last term on the right side of equation (4.1) can be written as

$$\begin{aligned}
\Delta t \sum_{j=0}^{3} b_j \left(\frac{\partial\vec{u}}{\partial t}\right)^{(n-j)} &= \Delta t b_0 \left(\frac{\partial\vec{u}}{\partial t}\right)^{(n)} \\
&+ \Delta t b_1 \left[\left(\frac{\partial\vec{u}}{\partial t}\right)^{(n)} - \left(\frac{\partial^2\vec{u}}{\partial t^2}\right)^{(n)} \Delta t + \frac{1}{2!}\left(\frac{\partial^3\vec{u}}{\partial t^3}\right)^{(n)} \Delta t^2 - \frac{1}{3!}\left(\frac{\partial^4\vec{u}}{\partial t^4}\right)^{(n)} \Delta t^3 + \ldots\right] \\
&+ \Delta t b_2 \left[\left(\frac{\partial\vec{u}}{\partial t}\right)^{(n)} - \left(\frac{\partial^2\vec{u}}{\partial t^2}\right)^{(n)} (2\Delta t) + \frac{1}{2!}\left(\frac{\partial^3\vec{u}}{\partial t^3}\right)^{(n)} (2\Delta t)^2 - \frac{1}{3!}\left(\frac{\partial^4\vec{u}}{\partial t^4}\right)^{(n)} (2\Delta t)^3 + \ldots\right] \\
&+ \Delta t b_3 \left[\left(\frac{\partial\vec{u}}{\partial t}\right)^{(n)} - \left(\frac{\partial^2\vec{u}}{\partial t^2}\right)^{(n)} (3\Delta t) + \frac{1}{2!}\left(\frac{\partial^3\vec{u}}{\partial t^3}\right)^{(n)} (3\Delta t)^2 - \frac{1}{3!}\left(\frac{\partial^4\vec{u}}{\partial t^4}\right)^{(n)} (3\Delta t)^3 + \ldots\right]
\end{aligned}$$

Substitute the above expression into equation (4.1), we have

$$\begin{aligned}
\frac{\vec{u}^{(n+1)} - \vec{u}^{(n)}}{\Delta t} &= [b_0 + b_1 + b_2 + b_3] \left(\frac{\partial\vec{u}}{\partial t}\right)^{(n)} \\
&+ [-b_1 - 2b_2 - 3b_3] \Delta t \left(\frac{\partial^2\vec{u}}{\partial t^2}\right)^{(n)} \\
&+ \left[\frac{b_1}{2!} + \frac{b_2}{2!}(2)^2 + \frac{b_3}{2!}(3)^2\right] \Delta t^2 \left(\frac{\partial^3\vec{u}}{\partial t^3}\right)^{(n)} \\
&+ \left[-\frac{b_1}{3!} - \frac{b_2}{3!}(2)^3 - \frac{b_3}{3!}(3)^3\right] \Delta t^3 \left(\frac{\partial^4\vec{u}}{\partial t^4}\right)^{(n)} + \mathcal{O}\left(\Delta t^4\right)
\end{aligned}$$

or

$$
\begin{aligned}
\vec{u}^{(n+1)} &= \vec{u}^{(n)} + \left[b_0 + b_1 + b_2 + b_3\right] \Delta t \left(\frac{\partial \vec{u}}{\partial t}\right)^{(n)} \\
&+ \left[-b_1 - 2b_2 - 3b_3\right] \Delta t^2 \left(\frac{\partial^2 \vec{u}}{\partial t^2}\right)^{(n)} \\
&+ \left[\frac{b_1}{2!} + \frac{b_2}{2!}(2)^2 + \frac{b_3}{2!}(3)^2\right] \Delta t^3 \left(\frac{\partial^3 \vec{u}}{\partial t^3}\right)^{(n)} \\
&+ \left[-\frac{b_1}{3!} - \frac{b_2}{3!}(2)^3 - \frac{b_3}{3!}(3)^3\right] \Delta t^4 \left(\frac{\partial^4 \vec{u}}{\partial t^4}\right)^{(n)} + \mathcal{O}\left(\Delta t^5\right)
\end{aligned}
$$

Compared to the Taylor series expansion for $\vec{u}^{(n+1)}$

$$
\vec{u}^{(n+1)} = \vec{u}^{(n)} + \Delta t \left(\frac{\partial \vec{u}}{\partial t}\right)^{(n)} + \frac{1}{2!}\Delta t^2 \left(\frac{\partial^2 \vec{u}}{\partial t^2}\right)^{(n)} + \frac{1}{3!}\Delta t^3 \left(\frac{\partial^3 \vec{u}}{\partial t^3}\right)^{(n)} + \frac{1}{4!}\Delta t^4 \left(\frac{\partial^4 \vec{u}}{\partial t^4}\right)^{(n)} + \mathcal{O}\left(\Delta t^5\right)
$$

we can set

$$
\sum_{j=0}^{3} b_j = 1 \qquad \sum_{j=1}^{3} b_j j = -\frac{1}{2}
$$

$$
\sum_{j=0}^{3} b_j j^2 = \frac{1}{3} \qquad \sum_{j=0}^{3} b_j j^3 = -\frac{1}{4}
$$

The four coefficients b_j of equation (4.1) can be completely determined. In order to construct an optimized multi-level scheme, we will choose three of the four coefficients b_j (e.g. j =1, 2, 3)

$$
\left.
\begin{aligned}
&\sum_{j=0}^{3} b_j = 1 \\
&\sum_{j=1}^{3} b_j j = -\tfrac{1}{2} \\
&\sum_{j=0}^{3} b_j j^2 = \tfrac{1}{3} \\
&\sum_{j=0}^{3} b_j j^3 = -\tfrac{1}{4}
\end{aligned}
\right\}
\qquad
\begin{aligned}
b_1 &= -3b_0 + \tfrac{53}{12} \\
b_2 &= 3b_0 - \tfrac{16}{3} \\
b_3 &= -b_0 + \tfrac{23}{12}
\end{aligned}
$$

The remaining coefficient b_0 will be determined by the optimization. The time discretization is therefore of 3^{rd} order.

The optimization is done by demanding the Laplace transform of the finite difference scheme (eq. (4.1)) to be a good approximation of that of the partial derivative. The Laplace transform and its inverse of a function $f(t)$ are related by

$$
\tilde{f}(\omega) = \frac{1}{2\pi} \int_{0}^{\infty} f(t)\, e^{i\omega t} dt
$$

$$
f(t) = \int_{\Gamma} \tilde{f}(\omega)\, e^{-i\omega t} d\omega
$$

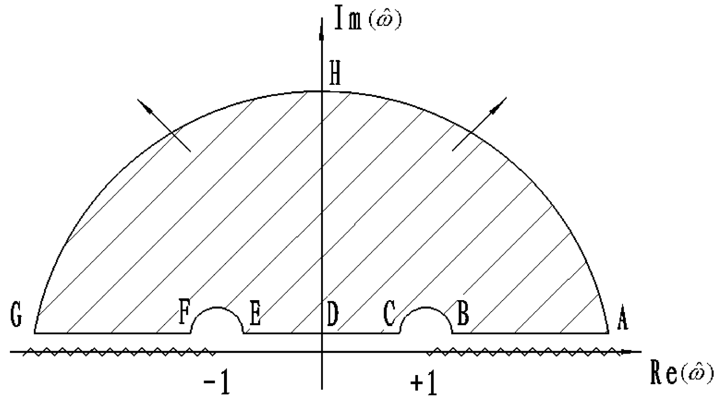


Figure 4.1: upper $\hat{\omega}$ half plane (Tam and Auriault [3])

The inverse contour Γ is a line in the upper half ω-plane parallel to the real-ω-axis above all poles and singularities.

Let us write equation (4.1) with a continuous variable

$$\vec{u}\,(t+\Delta t) = \vec{u}\,(t) + \Delta t \sum_{j=0}^{3} b_j \frac{\partial \vec{u}\,(t + j\Delta t)}{\partial t} \tag{4.2}$$

Applying the Laplace transform to equation (4.2) and using the shifting theorem

$$\tilde{f}\,(t+\Delta t) = e^{-i\omega\Delta t}\tilde{f}$$

we have

$$\begin{aligned} \tilde{u}e^{-i\omega\Delta t} &= \tilde{u} + \Delta t \left(\sum_{j=0}^{3} b_j e^{i\omega j\Delta t} \right) \frac{\partial \tilde{u}}{\partial t} \\ &\Rightarrow \frac{\tilde{u}\left[e^{-i\omega\Delta t} - 1\right]}{\Delta t \sum\limits_{j=0}^{3} b_j e^{i\omega j \Delta t}} = \frac{\partial \tilde{u}}{\partial t} \end{aligned}$$

With derivative theorem

$$\frac{\partial \tilde{u}}{\partial t} = -i\omega\tilde{u}$$

we have

$$\frac{e^{-i\omega\Delta t} - 1}{\Delta t \sum\limits_{j=0}^{3} b_j e^{i\omega j\Delta t}} = -i\omega$$

By comparing the two sides of the above equation we have

$$\overline{\omega} = \frac{i\left(e^{-i\omega\Delta t} - 1\right)}{\Delta t \sum\limits_{j=0}^{3} b_j e^{i\omega j\Delta t}}$$

where $\bar{\omega}$ is the effective angular frequency. The integrated error E_1 is then defined as

$$E_1 = \int_{-\eta}^{\eta} \left\{ \sigma \left[Re\left(\overline{\omega}\Delta t - \omega\Delta t\right)\right]^2 + (1-\sigma)\left[Im\left(\overline{\omega}\Delta t - \omega\Delta t\right)\right]^2 \right\} d\left(\omega\Delta t\right)$$

Where $\sigma \in [0;1]$ is the weighting factor and η is the frequency range we would like to have a good approximation between $\overline{\omega}$ and ω (similar to the optimization range δ for the spatial approximation). Therefore, the aim is to get $\bar{\omega} = \omega$. If $\sigma = 1$, only the real part of the deviation is considered, thus the dispersion. If $\sigma = 0$, only the imaginary part is considered, thus the dissipation.

To achieve a minimum integrated error, we need the condition for the undetermined coefficient b_0

$$\frac{\partial E_1}{\partial b_0} = 0$$

For $\sigma = 0.36$ and $\eta = 0.5$ we can determine the coefficients as

$$b_0 = 2.3025580888, \quad b_1 = -2.4910075998$$

$$b_2 = 1.5743409332, \quad b_3 = -0.3858914222$$

4.1 Group Velocity Consideration

Literature: Tam [4]

Let $A(x,t)$ be a fundamental solution of the (one-dimensional) wave equation

$$\frac{\partial^2 A}{\partial x^2} - \left(\frac{\alpha}{w}\right)^2 \frac{\partial^2 A}{\partial t^2} = 0$$

We have

$$A(x,t) = A_0 \cos(\alpha x - \omega t)$$

and the phase velocity

$$v_{ph} = \frac{\omega}{\alpha}$$

Consider a simple wave packet formed out of the superposition of two cosine waves

$$A(x,t) = \cos[(\alpha - \Delta\alpha)x + (\omega - \Delta\omega)t] + \cos[(\alpha - \Delta\alpha)x - (\omega - \Delta\omega)t]$$

Using a trigonometric identity

$$\begin{aligned}&\cos[(\alpha x - \omega t) \pm (\Delta\alpha x - \Delta\omega t)] \\ &= \cos[(\alpha x - \omega t)\cos(\Delta\alpha x - \Delta\omega t) \mp \sin(\alpha x - \omega t)\sin(\Delta\alpha x - \Delta\omega t)]\end{aligned}$$

Therefore

$$A(x,t) = 2\cos(\alpha x - \omega t)\cos(\Delta\alpha x - \Delta\omega t)$$

Now think of this wave packet as a cosine wave of frequency ω and wave number α modulated by a cosine function $\cos(\Delta\alpha x - \Delta\omega\, t)$. The modulated function is itself a wave, and the phase velocity of this modulation wave is $v = d\omega/d\alpha$.

E.g. $\omega = 6$ rad/sec $\quad\quad$ $\alpha = 6$ rad/meter
$d\omega = 0.1$ rad/sec $\quad\quad$ $d\alpha = 0.3$ rad/meter

The phase velocity of the internal oscillations is $v_{ph} = \omega/\alpha = 1$ meter/sec whereas the modulation envelope wave has a phase velocity $v_{gr} = d\omega/d\alpha = 0.33$ meter/sec.

The group velocity is the rate at which modulations within a wave travel through a given medium and information are transported.

For the DRP (Dispersion-Relation-Preserving) scheme, the dispersion relation of the wave is given as

$$\overline{\omega}\left(\omega\right) = \omega\left(\overline{\alpha}\left(\alpha\right)\right).$$

(Assuming waves propagate in the x-direction only) The group velocity of the wave is

$$v_{gr} = \frac{d\omega}{d\alpha} = \frac{d\omega}{d\overline{\omega}}\frac{d\overline{\omega}}{d\overline{\alpha}}\frac{d\overline{\alpha}}{d\alpha} = \frac{\frac{d\overline{\omega}}{d\overline{\alpha}}\frac{d\overline{\alpha}}{d\alpha}}{\frac{d\bar{\omega}}{d\omega}}$$

With an acceptable approximation $\frac{d\bar{\omega}}{d\omega} \simeq 1$ and from the numerical dispersion relation $\frac{d\bar{\omega}}{d\bar{\alpha}} \simeq c$ we obtain

$$v_{gr} \approx c\frac{d\bar{\alpha}}{d\alpha}$$

If $\frac{d\overline{\alpha}}{d\alpha} = 1$, the scheme is then to reproduce the same group velocity of the original partial differential equation.

Finally, the requirements on the numerical method for correct predictions should be $v_{gr} : \frac{d\bar{\alpha}}{d\alpha} \stackrel{!}{=} 1$ and $v_{ph} : \frac{\bar{\alpha}}{\alpha} \stackrel{!}{=} 1$.

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5 Review of The Basic- Governing Equations of Fluid Dynamics

The basic governing equations of fluid dynamics are

- Continuity (Conservation of mass)
- Momentum (Newton's second law)
- Energy (Conservation of energy, The first law of thermodynamic)

All the equations can be derived from the **Reynolds Transport Theorem**.

Consider a system and a control volume (C.V.) as follows:

- The system occupies C.V. (IrII) at time t
- The same system occupies C.V. (IIrIII) at time $t + \Delta t$
- The total rate of change of any extensive property B_{sys} of a system occupying a control volume at time t is equal to the sum of the following two terms
  (a) the temporal rate of change of B_{sys} within the C.V.
  (b) the net flux of B_{sys} through the control surface (C.S.) that surrounds the C.V.

The theorem can be applied to any transportable property, such as mass, momentum and energy.

$$\frac{dB_{sys}}{dt} = \frac{\partial}{\partial t}\int_{C.V.} b\rho dV + \int_{C.S.} b\rho \vec{u}\cdot\vec{n} dA$$

where $\vec{n}$ is a unit vector normal to the C.S. with positive pointing outward from the control surface.

From **Gauss Theorem**

$$\begin{aligned} \int_S \vec{a}\cdot\vec{n} dA &= \int_V \vec{\nabla}\cdot\vec{a} dV \\ \Rightarrow \int_{C.S.} b\rho\vec{u}\cdot\vec{n} dA &= \int_{C.V.} \vec{\nabla}\cdot(b\rho\vec{u})\, dV \end{aligned}$$

we then have

$$\frac{DB_{sys}}{Dt} = \frac{\partial}{\partial t}\int_{C.V.} b\rho dV + \int_{C.V.} \vec{\nabla}(b\rho\vec{u})\, dV$$

Conservation of mass

- $B_{sys} = m_{sys}$ with $\frac{Dm_{sys}}{Dt} = 0$
- $b = 1$
- $\Rightarrow 0 = \int_{C.V.} \frac{d\rho}{dt} + \int_{C.V} \vec{\nabla}\cdot(\rho\vec{u})\, dV$

The differential form of the conservation of mass

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{u}) = 0$$

Momentum (linear)

- $\overrightarrow{B}_{sys} = (m\vec{u})_{sys}$
- $\vec{b} = \vec{u}$

The differential form of the equation of motion

$$\rho \frac{d\vec{u}}{dt} = -\vec{\nabla} p + \vec{\nabla} \cdot \overline{\overline{\tau}} + \rho \vec{g}$$

where (Newtonian Viscous fluid)

$$\tau_{ij} = \mu \left[\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right] - \frac{2}{3} \mu \left(\vec{\nabla} \cdot \vec{u} \right) \delta_{ij}$$

($\overline{\overline{\tau}}$ is the viscous shear stress tensor)

For an inviscid fluid, we have the Euler's equation:

$$\rho \frac{d\vec{u}}{dt} = \rho \left(\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} \right) = -\vec{\nabla} p + \rho \vec{g}$$

Conservation of Energy

- $\vec{B}_{sys} = E_{sys}$
- $b = e$

The differential form of the energy e.g.

$$\rho \frac{d\hat{u}}{dt} + p \vec{\nabla} \cdot \vec{u} = \Phi + \vec{\nabla} \cdot \left(k \vec{\nabla} \tau \right) + g_H$$

where Φ is the dissipation function and g_H is the heat sources other than conduction (e.g. radiation, chemical reactions) and $\hat{u}$ is the internal energy per unit mass.

For an inviscid fluid flow without any heat sources, heat conduction and radiation, the energy equation can be deduced to

$$\rho \frac{d\hat{u}}{dt} + p \vec{\nabla} \cdot \vec{u} = 0$$

Assuming perfect gas adiabatic

$$p = \rho R T, \qquad p/\rho^{\gamma} = \text{ constant}$$

where $\gamma = c_p/c_V$ and $\hat{u} = c_V T$

The energy equation can be then written as

$$\frac{dp}{dt} + \gamma p \vec{\nabla} \cdot \vec{u} = 0 \text{ or}$$

$$\frac{d\rho}{dt} + \rho \vec{\nabla} \cdot \vec{u} = 0 \, *$$

$*$ The energy equation is identical to the continuity equation under the above assumptions

5.1 Introduction of The Acoustic Wave Equation

5.1.1 Terminology

Nomenclature

ρ	=	instantaneous density at (x, y, z)
ρ_0	=	equilibrium density at $\vec{x}$
s	=	condensation at $\vec{x}$, where $s = \rho/\rho_0 - 1$
$\rho - \rho_0$	=	$\rho_0 s$ = acoustic density at $\vec{x}$
p	=	instantaneous pressure at $\vec{x}$
p_0	=	equilibrium pressure at $\vec{x}$
p'	=	acoustic pressure at $\vec{x}$, where $p' = p - p_0$
c	=	thermodynamic speed of sound of the fluid.

The Equation of state:

$p = \rho RT$ (T in Kelvin)

with the gas constant $R = 287$ J/kgK for air.

For an isentropic process, we have

$$p/p_0 = (\rho/\rho_0)^\gamma, \qquad \gamma = c_p/c_V$$

where γ is the ration of specific heats.

For fluids other than a perfect gas, the isentropic relation between pressure and density fluctuations is determined preferably by experiments.

The relationship can also be expressed by a Taylor series expansion around the equilibrium state

$$p = p_0 + \left(\frac{\partial p}{\partial \rho}\right)_{\rho_0} (\rho - \rho_0) + \frac{1}{2}\left(\frac{\partial^2 p}{\partial \rho^2}\right)(\rho - \rho_0)^2$$

$$p - p_0 \approx \left(\frac{\partial p}{\partial \rho}\right)_{\rho_0} (\rho - \rho_0)$$

$$\Rightarrow p' \approx \rho_0 \left(\frac{\partial p}{\partial \rho}\right)_{\rho_0} \left[\frac{\rho}{\rho_0} - 1\right]$$

$$\Rightarrow p' \approx Bs$$

where $B = \rho_0 \left(\frac{\partial p}{\partial \rho}\right)_{\rho_0}$ is the adiabatic bulk modulus.

The thermodynamic speed of sound is defined by

$$c^2 = \frac{B}{\rho_0} = \left(\frac{\partial p}{\partial \rho}\right)_{\rho_0}$$

When a sound wave propagates through a perfect gas with adiabatic condition

$$\left(\frac{\partial p}{\partial \rho}\right)_{adiabat} = \frac{\partial}{\partial \rho}\left[p_0 \left(\frac{\rho}{\rho_0}\right)^\gamma\right] = \gamma p_0 \left(\frac{\rho}{\rho_0}\right)^\gamma \frac{1}{\rho} = \gamma \frac{p}{\rho} = \gamma RT$$

$$\boxed{c^2 = \gamma RT}$$

It is also noted for an incompressible fluid, we have $c \to \infty$.

5.1.2 Acoustic Field Equations

Since the disturbances associated with a sound wave are small and the propagation of a sound wave is nearly isentropic, the modeling of acoustic field can be based on the linearized governing equations with neglecting viscosity, heat condition and radiation (for a perfect gas).

Continuity: $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0$

Momentum: $\rho \frac{d\vec{u}}{dt} = -\nabla p \quad \text{or} \quad \rho \left[\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla} \right) \vec{u} \right] = -\nabla p$

Energy: $\frac{dp}{dt} + \gamma p \vec{\nabla} \cdot \vec{u} = 0$

Consider the small disturbances

$$\rho = \rho_0 + \rho'$$

$$p = p_0 + p'$$

$$\vec{u} = \vec{u}_0 + \vec{u'}$$

with ρ_0, p_0 and $\vec{u}_0$ constant in time and ρ_0 and p_0 constant in space.

After neglecting the second and higher order terms, the resulting acoustic field equations are given as the following:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{u'} + \rho_0 \vec{\nabla} \cdot \vec{u}_0 + \rho' \vec{\nabla} \cdot \vec{u}_0 + \left(\vec{u}_0 \cdot \vec{\nabla} \right) \rho' = 0$$

$$\frac{\partial \vec{u'}}{\partial t} + \left(\vec{u}_0 \cdot \vec{\nabla} \right) \vec{u'} + \left(\vec{u'} \cdot \vec{\nabla} \right) \vec{u}_0 + \frac{\rho_0 + \rho'}{\rho_0} \left(\vec{u}_0 \cdot \vec{\nabla} \right) \vec{u}_0 = -\frac{\nabla p'}{\rho_0}$$

$$\frac{\partial p'}{\partial t} + \left(\vec{u}_0 \cdot \vec{\nabla} \right) p' + \gamma p_0 \vec{\nabla} \cdot \vec{u}_0 + \gamma p' \vec{\nabla} \cdot \vec{u}_0 + \gamma p_0 \vec{\nabla} \cdot \vec{u'} = 0$$

The above three equations are called acoustic continuity, momentum and energy equations (or acoustic field equations). It can be shown easily that the above equations will remain unchanged in a non-dimensional form if the characteristic variables are given as follows (r=reference state):

$\rho \rightarrow \rho_r$

$\vec{u} \rightarrow \vec{c}_r$ (speed of sound)

$p \rightarrow \rho_r c_r^2$

$t \rightarrow L/c_r$ (L is the reference length)

If there is no mean flow ($\vec{u}_0 = 0$) the acoustic field equations are

1. $\frac{\partial \rho'}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{u'} = 0$
2. $\frac{\partial \vec{u'}}{\partial t} + \frac{1}{\rho_0} \nabla p' = 0$
3. $\frac{\partial p'}{\partial t} + \gamma p_0 \left(\vec{\nabla} \cdot \vec{u'} \right) = 0$

Using the isentropic relation $p' = c^2 \rho'$ one can show that Eq (1.) and Eq (3.) are identical. Thus there exists an algebraic relation between pressure and density which results in timesaving computation. This is true if the average flow is constant or for a potential flow and if the disturbances are adiabatic (like in acoustics). In general in the linearized energie equation is an non-isentropic relation between pressure and density (p is independet of ρ)

Taking the time derivative of Eq (3.)

$$\frac{\partial^2 p'}{\partial t^2} + \gamma p_0 \frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{u'}\right) = 0$$

and subtracting $\gamma p_0 \nabla \cdot$Eq (2.)

$$-\gamma p_0 \left[\vec{\nabla} \cdot \left(\frac{\partial \vec{u'}}{\partial t}\right) + \frac{1}{\rho_0}\left(\vec{\nabla} \cdot \vec{\nabla}\right) p'\right] = 0$$

we then have

$$\frac{\partial^2 p'}{\partial t^2} = c_0^2 \left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} + \frac{\partial^2 p'}{\partial z^2}\right) \text{ or}$$

$$\boxed{\frac{\partial^2 p'}{\partial t^2} = c_0^2 \Delta p'}$$

the standard second-order scalar wave equation or the linear wave equation. The wave equation holds true for ρ', τ' and $\vec{\nabla} \cdot \vec{u}$, given the assumption that the ambient medium is homogeneous and quiescent.

Introducing $\vec{u} = \nabla \Phi$ (velocity potential) with $\underline{\nabla} \times \vec{u'} = 0$ and $\underline{\nabla} \times \vec{u}_0 = 0$, it can be proved that

$$\frac{\partial^2 \Phi}{\partial t} = c_0^2 \nabla^2 \Phi$$

However, the potential equation only describes the acoustics and not moving vortices ($\underline{\nabla} \times \vec{u'} \neq 0$) or non-isentropic disturbances ($s' \neq 0$).

5.1.3 Harmonic Plane Waves

Discussions will be restricted to $^{1)}$homogeneous, $^{2)}$isentropic fluids and $^{3)}$speed of sound c is a constant throughout.

Definition of plane waves:
All acoustic field qualities vary with time and with some Cartesian coordinate s only and these quantities are independent of position along plane normal to the s-direction.

Therefore we have

$$\begin{aligned} p &= p(s,t) \\ \vec{u} &= u(s,t)\,\vec{n} \end{aligned}$$

where $\vec{n}$ is the unit vector in the directions of s.

For a plane wave traveling in an arbitrary direction a solution can be written as

$$p = Ae^{i(\omega t - k_x x - k_y y - k_z z)}$$

Substitution the solution into the wave equation

$$\frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p$$

we have magnitude

$$\left(\frac{\omega}{c}\right)^2 = k_x^2 + k_y^2 + k_z^2 \qquad \text{or} \qquad k = \frac{\omega}{c}$$

Define the wave propagation vector $\vec{k}$.

$$\vec{k} = k_x\hat{x} + k_y\hat{y} + k_z\hat{z}$$

with a position vector $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$ we have $p = Ae^{i\left(\omega t - \vec{k}\cdot\vec{r}\right)}$ The surfaces of constant phase are given by $\vec{k}\cdot\vec{r} =$ constant. As a special case, we are going to examine a plane wave whose surfaces of constant phase are parallel to the z-axis.

$$p = Ae^{i(\omega t - k_x x - k_y y)}$$

The surfaces of constant phase are given by

$k_x x + k_y y =$ constant
or
$y = -\left(\frac{k_x}{k_y}\right) x =$ constant.

which describes plane surfaces parallel to the z-axis with a slope of $-\frac{k_x}{k_y}$ in the $x = y$ plane. The vector $\vec{k}$ is perpendicular to the z-axis.

$$\vec{k} = k_x \cos\varphi\hat{x} + k_y \sin\varphi\hat{y}$$

$\vec{k}$ points in the direction of propagation.

The magnitude of $\vec{k}$ is the wave number k and $\frac{k_x}{k} = \cos\varphi$, $\frac{k_y}{k} = \sin\varphi$. The wave length λ is defined as (wavelength is distance sound travels in once cycle)

$$\lambda = \frac{2\pi}{k}$$

since we have $k = \frac{\omega}{c}$ and $\omega = 2\pi f$

$$f = \frac{\omega}{2\pi}$$

- ω is the angular frequency
- f is the number of cycles per unit time (frequency)
- the units of frequency are hertz (Hz), where $1Hz$ equals 1 cycle per second

The wave length can be derived as

$$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} = \frac{c}{f}$$

$$\boxed{\lambda f = c}$$

The speed of sound in air is approximately $340m/s$. The wavelength corresponding to a frequency of $262Hz$ (middle C on the piano) is $1,3m$. Therefore typical sound wavelengths are neither too long more too short considering the human dimensions. Frequencies audible to a normal human ear are roughly between 20 to $20000Hz$.

$f = 20Hz$	$\rightarrow$	$\lambda = 17m$
$f = 20000Hz$	$\rightarrow$	$\lambda = 0.017m$

A young person can detect pressure as low as about $20\mu Pa$ compared to the normal atmospheric pressure ($101,3 \times 10^3 Pa$) around which it varies, a fractional variation of 2×10^{-10}.

6 Finite Difference Solution of the Linearized Euler Equations

6.1 The conservative form of a PDE

Definition:
The coefficients of the derivatives are either constant or, if variable, their derivatives don't appear anywhere in the equation. For example, the conservation of mass for steady two-dimensional flow is written in a conservative form as

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \tag{6.1}$$

and is written in a non-conservative form as

$$\rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} = 0 \tag{6.2}$$

6.2 The conservative form of the Euler Equations

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} + \frac{\partial G}{\partial z} = 0 \tag{6.3}$$

where

$$Q = \begin{bmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ \rho e_t \end{bmatrix}, E = \begin{bmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ (\rho e_t + p)\, u \end{bmatrix}, F = \begin{bmatrix} \rho v \\ \rho vu \\ \rho v^2 + \rho \\ \rho vw \\ (\rho e_t + p)\, v \end{bmatrix}, G = \begin{bmatrix} \rho w \\ \rho wu \\ \rho wv \\ \rho w^2 + p \\ (\rho e_t + p)\, w \end{bmatrix}$$

and

$$e_t = e + \frac{1}{2}\left(u^2 + v^2 + w^2\right)$$

Consider small amplitude disturbances superimposed on a uniform mean flow with density ρ_0, pressure p_0 and velocity u_0 in the x-direction only. The linearized Euler equations for two-dimensional disturbances are (assuming $p = \rho RT$ and $p/\rho^\gamma =$ constant)

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = H \tag{6.4}$$

where

$$Q = \begin{bmatrix} \rho' \\ u' \\ v' \\ p' \end{bmatrix}, E = \begin{bmatrix} \rho_0 u' + \rho' u_0 \\ u_0 u' + p'/\rho_0 \\ u_0 v' \\ u_0 p' + \gamma p_0 u' \end{bmatrix}, F = \begin{bmatrix} \rho_0 v' \\ 0 \\ p'/\rho_0 \\ \gamma p_0 v' \end{bmatrix}$$

The non-homogeneous term H represents sources. Equation (6.4) can also be written as

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial Q}\frac{\partial Q}{\partial x} + \frac{\partial F}{\partial Q}\frac{\partial Q}{\partial y} = H$$

Therefore

$$\rightarrow \frac{\partial Q}{\partial t} + \begin{bmatrix} u_0 & \rho_0 & 0 & 0 \\ 0 & u_0 & 0 & \frac{1}{\rho_0} \\ 0 & 0 & u_0 & 0 \\ 0 & \gamma p_0 & 0 & u_0 \end{bmatrix} \frac{\partial Q}{\partial x} + \begin{bmatrix} 0 & 0 & \rho_0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\rho_0} \\ 0 & 0 & \gamma p_0 & 0 \end{bmatrix} \frac{\partial Q}{\partial y} = H \tag{6.5}$$

6.3 Wave Decomposition

The Fourier-Laplace transform of a function $f(x, y, t)$ and its inverse are defined as

$$\begin{aligned}\tilde{f}(\alpha,\beta,\omega) &= \frac{1}{(2\pi)^3}\int_0^\infty\int_{-\infty}^\infty\int_{-\infty}^\infty f(x,y,t)\, e^{-i(\alpha x+\beta y-\omega t)}dxdydt \\ f(x,y,t) &= \int_\Gamma\int_{-\infty}^\infty\int_{-\infty}^\infty \tilde{f}(\alpha,\beta,\omega)e^{i(\alpha x+\beta y-\omega t)}d\alpha d\beta d\omega\end{aligned}$$

The contour Γ is a line parallel to the real axis in the complex ω-plane above all poles and singularities of the integrand.

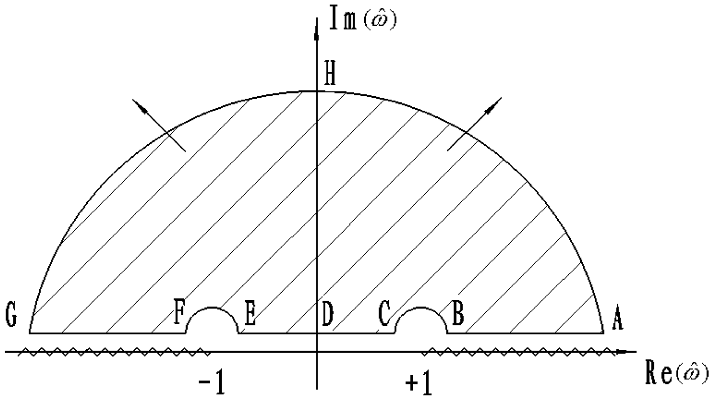


Figure 6.1: upper $\hat{\omega}$ half plane (Tam and Auriault [1])

The general initial value problem can be solved

$$\frac{1}{2\pi}\int_0^\infty \frac{\partial \mathbf{u}}{\partial t}e^{i\omega t}dt = -\frac{1}{2\pi}\mathbf{u}_{\text{initial}} - i\omega\tilde{\mathbf{u}}$$

Therefore the Fourier-Laplace transform of equation (6.5) can be written as

$$A\tilde{Q} = i\left(\tilde{H} + \frac{Q_{\text{initial}}}{2\pi}\right) \tag{6.6}$$

where

$$A = \begin{bmatrix} \omega-\alpha u_0 & -\rho_0\alpha & -\rho_0\beta & 0 \\ 0 & \omega-\alpha u_0 & 0 & -\alpha/\rho_0 \\ 0 & 0 & \omega-\alpha u_0 & -\beta/\rho_0 \\ 0 & -\gamma p_0\alpha & -\gamma p_0\beta & \omega-\alpha u_0 \end{bmatrix}$$

Details

$$\frac{\partial\tilde{\rho}}{\partial t} + u_0\frac{\partial\tilde{\rho}}{\partial x} + \rho_0\frac{\partial\tilde{u}}{\partial x} + \rho_0\frac{\partial\tilde{v}}{\partial y} = \tilde{H}_1$$

$$\begin{aligned}\rightarrow \quad & -\tfrac{1}{2\pi}\rho_{\text{initial}} - i\omega\tilde{\rho} + u_0 i\alpha\tilde{\rho} + \rho_0 i\alpha\tilde{u} + \rho_0 i\beta\tilde{v} = \tilde{H}_1 \\ & -i\left[(\omega-\alpha u_0)\tilde{\rho} - \rho_0\alpha\tilde{u} - \rho_0\beta\tilde{v}\right] = \tfrac{1}{2\pi}\rho_{\text{initial}} + \tilde{H}_1 \\ & (\omega-\alpha u_0)\tilde{\rho} - \rho_0\alpha\tilde{u} - \rho_0\beta\tilde{v} = i\left(\tilde{H}_1 + \tfrac{\rho_{\text{initial}}}{2\pi}\right)\end{aligned}$$

The eigenvalues λ_j and eigenvectors $\vec{x}_j$ $(j = 1, 2, 3, 4)$ of matrix A are $A\vec{x} = \lambda\vec{x}$ $\Rightarrow \det(A - \lambda I_n) = 0$. We have

$$\begin{aligned} \lambda_1 = \lambda_2 &= (\omega - \alpha u_0) \\ \lambda_3 &= (\omega - \alpha u_0) + c_0\left(\alpha^2 + \beta^2\right)^{\frac{1}{2}} \\ \lambda_4 &= (\omega - \alpha u_0) - c_0\left(\alpha^2 + \beta^2\right)^{\frac{1}{2}} \end{aligned}$$

where $c_0 = \sqrt{\gamma \frac{p_0}{\rho_0}}$ is the speed of sound. The eigenvectors are

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 0 \\ \beta \\ -\alpha \\ 0 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} \frac{1}{c_0^2} \\ \frac{-\alpha}{\rho_0 c_0\sqrt{\alpha^2+\beta^2}} \\ \frac{-\beta}{\rho_0 c_0\sqrt{\alpha^2+\beta^2}} \\ 1 \end{bmatrix}, \quad \vec{x}_4 = \begin{bmatrix} \frac{1}{c_0^2} \\ \frac{\alpha}{\rho_0 c_0\sqrt{\alpha^2+\beta^2}} \\ \frac{\beta}{\rho_0 c_0\sqrt{\alpha^2+\beta^2}} \\ 1 \end{bmatrix}$$

Equation (6.6) can be written as

$$A\tilde{Q} = \tilde{T} \qquad \text{where} \qquad \tilde{T} = i\left(\tilde{H} + \frac{Q_{\text{initial}}}{2\pi}\right)$$

Since the matrix A can be diagonalized and written as

$$A = X\Lambda X^{-1}$$

We then have

$$X\Lambda X^{-1}\tilde{Q} = \tilde{T}$$

where X and Λ are eigenvector matrix and eigenvalue matrix.

$$x = [x_1 x_2 x_3 x_4]$$

$$\Lambda = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{bmatrix}$$

$$\Rightarrow \quad \tilde{Q} = \frac{c_1}{\lambda_1}\vec{x}_1 + \frac{c_2}{\lambda_2}\vec{x}_2 + \frac{c_3}{\lambda_3}\vec{x}_3 + \frac{c_4}{\lambda_4}\vec{x}_4 \tag{6.7}$$

where $\vec{c}$ is a coefficient vector and is given by

$$\vec{c} = x^{-1}\tilde{T}$$

Equation (6.7) represents the decomposition of the solution into the entropy wave $\vec{x}_1$, the vorticity wave $\vec{x}_2$, and two modes of acoustic waves, $\vec{x}_3$ and $\vec{x}_4$.

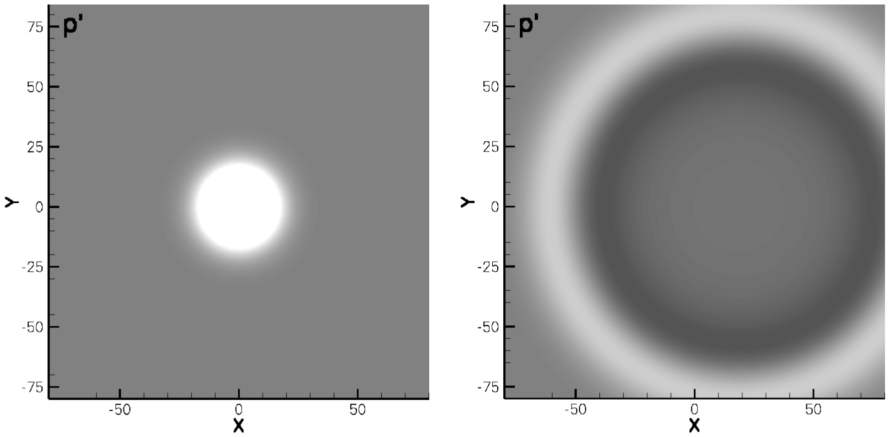
Details

$$\begin{aligned} X\Lambda X^{-1}\tilde{Q} &= \tilde{T} \\ X^{-1}X\Lambda X^{-1}\tilde{Q} &= X^{-1}\tilde{T} \\ \Lambda X^{-1}\tilde{Q} &= \vec{c} \\ \Lambda^{-1}\Lambda X^{-1}\tilde{Q} &= \Lambda^{-1}\vec{c} \end{aligned}$$

$$X^{-1}\tilde{Q} = \begin{bmatrix} \frac{1}{\lambda_1} & & & \\ & \frac{1}{\lambda_2} & & \\ & & \frac{1}{\lambda_3} & \\ & & & \frac{1}{\lambda_4} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} \frac{c_1}{\lambda_1} \\ \frac{c_2}{\lambda_2} \\ \frac{c_3}{\lambda_3} \\ \frac{c_4}{\lambda_4} \end{bmatrix} \Rightarrow \tilde{Q} = X \begin{bmatrix} \frac{c_1}{\lambda_1} \\ \frac{c_2}{\lambda_2} \\ \frac{c_3}{\lambda_3} \\ \frac{c_4}{\lambda_4} \end{bmatrix} = [\vec{x}_1\vec{x}_2\vec{x}_3\vec{x}_4] \begin{bmatrix} \frac{c_1}{\lambda_1} \\ \frac{c_2}{\lambda_2} \\ \frac{c_3}{\lambda_3} \\ \frac{c_4}{\lambda_4} \end{bmatrix}$$

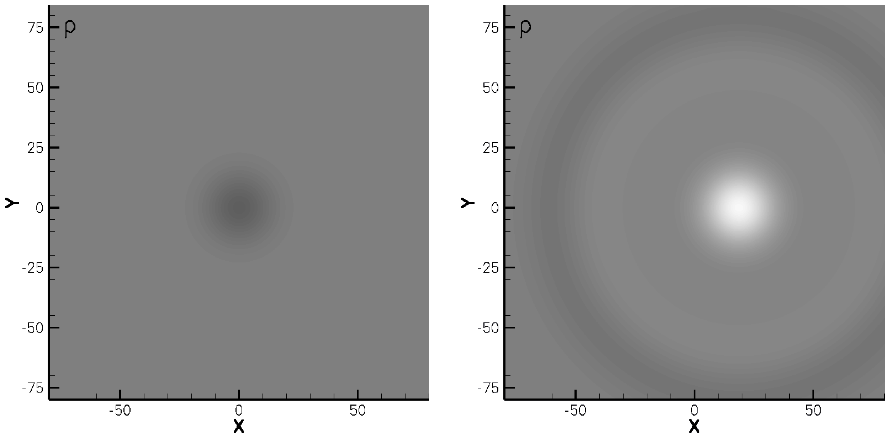
6.4 Definitions

1. The entropy wave: it consists of density fluctuations alone. i.e., $u' = v' = p' = 0$
2. The vorticity wave: it consists of velocity fluctuations alone. i.e., $p' = \rho' = 0$ (no pressure and density fluctuations associated with this wave mode)
3. The acoustic wave: it involves fluctuations of all the physical variables



initial condition: pressure

later



initial condition: density

later

1. homogeneous medium: any mean flow values are independent of $\vec{x}$
   waves propagate relatively to the constant flow
2. homentropic medium: (potential flow)
   entropy, vorticity and acoustic modes coupled through mean flow
3. quiescent medium: homentropic and homogeneous, $\vec{u}_0 = 0$

6.5 Propagation Speeds of the Waves

6.5.1 The Entropy Wave

$$\tilde{\rho} = \frac{c_1}{\lambda_1}$$

By the inverse Fourier-Laplace transform,

$$\rho'(x,y,t) = \int_\Gamma \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int \frac{c_1}{(\omega - \alpha u_0)} e^{-i(\alpha x + \beta y - \omega t)} d\alpha d\beta d\omega \qquad (6.8)$$

If $\omega - \alpha u_0 = 0$ (zero of the denominator), equation (6.8) gives rise to a pole of the integrand. The dispersion relation arising from this zero is

$$\lambda_1 = (\omega - \alpha u_0) = 0$$

In the α-plane, the zero of equation (6.8) is given by

$$\alpha = \frac{\omega}{u_0}$$

Using the residue theorem and Jordan's Lemma, the α-integral of equation (6.8) can be evaluated and we have

$$\rho'(x,y,t) = \begin{cases} 2\pi i \int_\Gamma \int_\beta \frac{c_1 e^{i\left(\frac{x}{u_0} - t\right)\omega + i\beta y}}{u_0} d\omega d\beta & x \to \infty \\ 0 & x \to -\infty \end{cases} \qquad (6.9)$$

or

$$\rho'(x,y,t) = \begin{cases} f(x - u_0 t, y) & x \to \infty \\ 0 & x \to -\infty \end{cases}$$

The entropy wave is convected downstream at the mean flow speed!

6.5.2 The Vorticity Wave

By applying the inverse Fourier-Laplace transforms, we have

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \int_\Gamma \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \begin{bmatrix} \beta \\ -\alpha \end{bmatrix} \frac{c_2}{(\omega - \alpha u_0)} e^{i(\alpha x + \beta y - \omega t)} d\alpha d\beta d\omega$$

The dispersion relation is

$$\lambda_2 = \omega - \alpha u_0 = 0.$$

Let

$$\psi(x,y,t) = \int_\Gamma \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{-ic_2}{(\omega - \alpha u_0)} e^{i(\alpha x + \beta y - \omega t)} d\alpha d\beta d\omega \qquad (6.10)$$

then

$$u' = \frac{\partial \psi}{\partial y} \qquad \text{and} \qquad v' = -\frac{\partial \psi}{\partial y}.$$

Equation (6.10) can be evaluated in the same way as equation (6.8). This gives us

$$\psi(x,y,t) = \begin{cases} \psi(x - u_0 t, y) & x \to \infty \\ 0 & x \to -\infty \end{cases} \qquad (6.11)$$

Like the entropy wave, the vorticity wave is convected downstream as at the mean flow speed.

6.5.3 The Acoustic Wave

The acoustic waves involve fluctuations in all the physical variables. The dispersion relation is given by

$$\lambda_3\lambda_4 = (\omega - au_0)^2 - c_0^2\left(\alpha^2 + \beta^2\right) = 0.$$

Applying the inverse Fourier-Laplace transforms and following a similar procedure as before, we have the asymptotic solution ($r \to \infty$)

$$\begin{bmatrix} \rho' \\ u' \\ v' \\ p' \end{bmatrix} \sim \frac{F\left(\frac{r}{V(\theta)} - t, \theta\right)}{\sqrt{r}} \cdot \begin{bmatrix} \frac{1}{c_0^2} \\ \frac{\hat{u}(\theta)}{\rho_0 c_0} \\ \frac{\hat{v}(\theta)}{\rho_0 c_0} \\ 1 \end{bmatrix} + \mathcal{O}\left(\sqrt{r}\right) \tag{6.12}$$

where $V(\theta)$ is the effective velocity of propagation in the θ-direction, and (r, θ) are the polar coordinates and

$$V(\theta) = u_0\cos\theta + c_0\sqrt{1 - M^2\sin^2\theta} \qquad M = \frac{u_0}{c_0}$$

$$\hat{u}(\theta) = \frac{\cos\theta - M\sqrt{1 - M^2\sin^2\theta}}{\sqrt{1 - M^2\sin^2\theta}} - M\cos\theta \quad \hat{V}(\theta) = \sin\theta\left[\sqrt{1 - M^2\sin^2\theta} + M\cos\theta\right]$$

6.5.4 Another Approach on Wave Decomposition

(In time domain, one-dimensional uniform flow) The Euler equation can be written as

$$\frac{\partial\vec{Q}}{\partial t} + A_0\frac{\partial\vec{Q}}{\partial x} = 0 \tag{6.13}$$

where

$$\vec{Q} = \begin{bmatrix} \rho' \\ u' \\ p' \end{bmatrix} \quad \text{and} \quad A_0 = \begin{bmatrix} u_0 & \rho_0 & 0 \\ 0 & u_0 & \frac{1}{\rho_0} \\ 0 & \gamma p & u_0 \end{bmatrix}$$

The matrix A_0 can be diagonalized as

$$A_0 = T\Lambda T^{-1} = \begin{bmatrix} 1 & \frac{\rho_0}{\sqrt{2}c_0} & \frac{\rho_0}{\sqrt{2}c_0} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{\rho_0 c_0}{\sqrt{2}} & \frac{\rho_0 c_0}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} u_0 & 0 & 0 \\ 0 & u_0 + c_0 & 0 \\ 0 & 0 & u_0 - c_0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -\frac{1}{c_0^2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}\rho_0 c_0} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}\rho_0 c_0} \end{bmatrix}$$

We can project the disturbance vector $\vec{Q}$ on the three independent eigenvectors and write

$$\vec{Q} = s_1\vec{x}_1 + s_2\vec{x}_2 + s_3\vec{x}_3 \tag{6.14}$$

where s_1, s_2 and s_3 are the three projections in the three directions.

Substitute equation (6.14) into equation (6.13) and consider

$$\begin{aligned} A_0\frac{\partial\vec{Q}}{\partial x} &= A_0\frac{\partial s_1\vec{x_1}}{\partial x} + A_0\frac{\partial s_2\vec{x_2}}{\partial x} + A_0\frac{\partial s_3\vec{x_3}}{\partial x} \\ &= \frac{\partial s_1\lambda_1\vec{x_1}}{\partial x} + \frac{\partial s_2\lambda_2\vec{x_2}}{\partial x} + \frac{\partial s_3\lambda_3\vec{x_3}}{\partial x} \end{aligned}$$

we have

$$\begin{aligned}
\frac{\partial}{\partial t}(s_1\vec{x_1}) &+\lambda_1 \tfrac{\partial}{\partial x}(s_1\vec{x_1}) &+ \\
\frac{\partial}{\partial t}(s_2\vec{x_2}) &+\lambda_2 \tfrac{\partial}{\partial x}(s_2\vec{x_2}) &+ \\
\frac{\partial}{\partial t}(s_3\vec{x_3}) &+\lambda_3 \tfrac{\partial}{\partial x}(s_3\vec{x_3}) &= 0
\end{aligned}$$

where $\lambda_1 = u_0$, $\lambda_2 = u_0 + c_0$, $\lambda_3 = u_0 + c_0$.

Therefore the disturbance vector is decomposed into three vectors what are propagating with speeds of u_0, $u_0 + c_0$ and $u_0 - c_0$, respectively. As it is shown that the entropy wave is traveling with the flow, and the acoustic waves are propagating with the speed of sound relative to the flow. Similar analogy can be made to the eigenvalues and eigenvectors of the Jacobian matrix (A_0) in multi-dimension.

6.5.5 Yet another Approach: solution modes by Kovasznay

The acoustic, entropy and vorticity modes are decoupled in terms of linearization. However, they are coupled of first order in non-constant flows.

Entropy Mode Consider a constant flow in x-direction ($\vec{u}_0 = u_0$). The energy equation can then be written as

$$\frac{\partial s}{\partial t} + \vec{u} \cdot \vec{\nabla} s = \vec{0}$$

Linearizataion ($s = s_0 + s'$) leads to

$$\frac{\partial (s_0 + s')}{\partial t} + (\vec{u}_0 + \vec{u'}) \cdot \vec{\nabla}(s_0 + s') = \vec{0}$$

and with s_0 being constant in time and space

$$\vec{0} = \frac{\partial s'}{\partial t} + \vec{u}_0 \cdot \vec{\nabla} s' + \mathcal{O}(\vec{u'}s') \approx \frac{Ds'}{Dt} \tag{6.15}$$

In 1D, the approach $s' = f(x - u_0 t)$ satisfies this equation. This means, that the entropy mode behaves like acoustic waves (the shape of the entropy wave remains constant) but with a propagation speed of u_0 instead of c_0.

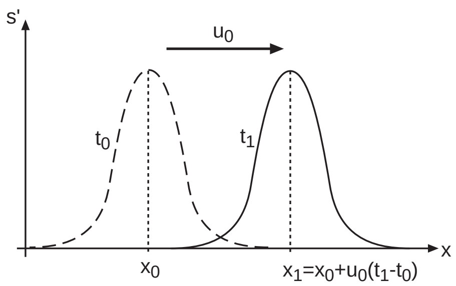


Figure 6.2: propagation of an entropy wave in 1D

In 2D, the approach $s' = f(x - u_0 t, y)$ satisfies equation (6.15).

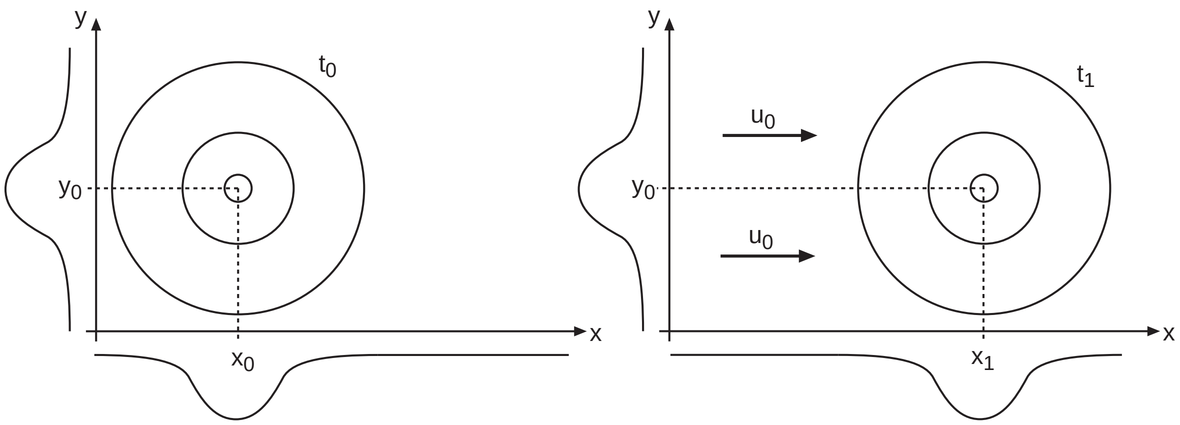


Figure 6.3: propagation of an entropy wave in 2D

Thus, the hot-spot propagates - different from acoustic waves - only in direction of the mean flow. The entropy wave can be "switched off" by using the isentropic relation $p' = c^2\rho'$ instead of solving $\frac{Ds'}{Dt}$.

Vorticity Mode The momentum equation with $\vec{u}_0, \rho_0 = \text{const.}$ is

$$\frac{\partial \vec{u'}}{\partial t} + \vec{u}_0 \cdot \vec{\nabla}\vec{u'} + \frac{1}{\rho_0}\vec{\nabla}p' = 0$$

By applying rotation to this equation one gets

$$\frac{\partial \vec{\nabla} \times \vec{u'}}{\partial t} + \nabla \times (\vec{u}_0 \cdot \vec{\nabla}\vec{u'}) + \frac{1}{\rho_0}\vec{\nabla} \times (\vec{\nabla}p') = 0$$

and with $\vec{\nabla} \times \vec{\nabla} = \text{rotgrad} = 0$ as well as $\vec{u}_0 \cdot \vec{\nabla}\vec{u'} = \vec{\nabla}(\vec{u}_0 \cdot \vec{u'}) - \vec{u}_0 \times (\vec{\nabla} \times \vec{u'})$ and $\vec{\Omega'} = \vec{\nabla} \times \vec{u'}$

$$\frac{\partial \vec{\Omega'}}{\partial t} + \underbrace{\vec{\nabla} \times \vec{\nabla}(\vec{u}_0 \cdot \vec{u'})}_{=0} - \vec{\nabla} \times (\vec{u}_0 \times \vec{\Omega'}) = 0$$

Using the transformation $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ on the last term leads to

$$\vec{\nabla} \times (\vec{u}_0 \times \vec{\Omega'}) = \vec{u}_0(\vec{\nabla} \cdot \vec{\Omega'}) - \vec{\Omega'} \underbrace{(\vec{\nabla} \cdot \vec{u}_0)}_{=0 \text{ for } \vec{u}_0 = \text{const.}}$$

$$\Rightarrow \frac{\partial \vec{\Omega'}}{\partial t} - \vec{u}_0 \vec{\nabla} \cdot \vec{\Omega'} = 0 \qquad (6.16)$$

This equation is similar to the entropy equation (6.15). So, Solutions of equation (6.16) are $\vec{\Omega'} = f(x - \vec{u}_0 t)$. Vorticity waves are therefore propagating with the speed of the mean flow $\vec{u}_0$ whereas acoustic waves propagete with $\vec{u}_0 \pm c$. As the energy equation describes the transportation of entropy, the momentum equation describes the transportation of vortices. These two modes are called *hydrodynamic perturbations*. The missing acoustic waves don't describe the transport of vortices and entropy.

For an entropy perturbation with a frequency of $f = 100Hz$ and a flow speed of $u_0 = 10m/s$, the wavelength of the entropy wave is

$$\lambda_s = \frac{u_0}{f} = 0.1m$$

for $f = 1kHz$: $\lambda_s = 0.01m$ and for $f = 10kHz$: $\lambda_s = 0.001m$. To resolve these waves, the grid distance must be at least $0.001m$.

Notes

- The hydrodynamic perturbations are not small in terms of acoustics.
- For every frequency: $\lim_{u_0 \to 0} \vec{u}_0 \cdot \vec{\nabla} s' = \infty$ (similar for $\vec{\Omega}'$). Therefore, linearization is limited reasonable. Non-linear interaction of 2^{nd} order are to be expected. Leaving out the hydrodynamic perturbations is preferable. In practice, only some terms are masked like $\frac{\partial u'}{\partial y}$ or $\frac{\partial v'}{\partial x}$

References

[1] C. K. W. Tam and L. Auriault. Time-Domain Impedance Boundary Conditions for Computational Aeroacoustics. *AIAA Journal*, 34(5):917–923, May 1996.

7 Solve the Linearized Euler Equation using DRP Schemes

The linearized Euler equation (two-dimension)

$$\frac{\partial Q}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial y} = H \tag{7.1}$$

Where

$$Q = \begin{bmatrix} \rho' \\ u' \\ v' \\ p' \end{bmatrix}, \quad E = \begin{bmatrix} \rho_0 u' + \rho' u_0 \\ u_0 u' + p'/\rho_0 \\ u_0 v' \\ u_0 p' + \gamma p_0 u' \end{bmatrix}, \quad F = \begin{bmatrix} \rho_0 v' \\ 0 \\ p'/\rho_0 \\ \gamma p_0 v' \end{bmatrix}$$

Rewrite the equation (7.1) in the form, we have

$$I: \quad K = -\tfrac{\partial E}{\partial x} - \tfrac{\partial F}{\partial y} + H \tag{7.2}$$

$$II: \qquad \tfrac{\partial Q}{\partial t} = K \tag{7.3}$$

Using the 4^{th}-order 7-point optimized central finite difference scheme for the spatial discretization and the 3^{rd}-order 4 level optimized time marching scheme, the finite difference equation can be written as

$$I: \quad K_{l,m}^{(n)} = -\tfrac{1}{\Delta x} \sum_{j=-3}^{3} a_j E_{l+j,m}^{(n)} - \tfrac{1}{\Delta y} \sum_{j=-3}^{3} a_j F_{l,m+j}^{(n)} + H_{l,m}^{(n)} \tag{7.4}$$

$$II: \qquad Q_{l,m}^{(n+1)} = Q_{l,m}^{(n)} + \Delta t \sum_{j=0}^{3} b_j K_{l,m}^{(n-j)} \tag{7.5}$$

where l, m and n are the x, y, t indices and Δx, Δy, Δt are the mesh sizes and the time step. To reduce numerical truncation errors, it's best practice to first summarize the small coefficients ($a_{\pm 3}$, b_{-3}) and afterwards the bigger ones ($a_{\pm 1}$, b_0).

From the initial conditions

$$Q(x, y, 0) = Q_{\text{initial}}(x, y)$$

we have

$$Q_{l,m}^{(0)} = Q_{\text{initial}}$$

How about $Q_{l,m}^{(n)}$ for negative values of n ($n = -1$, -2 and -3)?
We will show later that the proper values for $Q_{l,m}^{(n)}$ are

$$Q_{l,m}^{(n)} = 0 \qquad \text{for negative values of } n$$

Therefore, for $n < 0$ set either $K_{l,m}^{(n)} = 0$ or $K_{l,m}^{(n)} = K_{l,m}^{(0.)}$

7.1 Question

Will the numerical scheme (equations (7.4) and (7.5)) give the same dispersion relations as the original partial differential equations (equations (7.2) and (7.3))?
In order to answer this question, we need to apply the Fourier-Laplace transforms to the finite difference equations with continuous variables:

$$K(x,y,t) = -\frac{1}{\Delta x}\sum_{j=-3}^{3} a_j E(x+j\Delta x,y,t) - \frac{1}{\Delta y}\sum_{j=-3}^{3} a_j F(x,y+j\Delta y,t) + H(x,y,t) \tag{7.6}$$

$$Q(x,y,t+\Delta t) = Q(x,y,t) + \Delta t\sum_{j=0}^{3} b_j K(x,y,t-j\Delta t) \tag{7.7}$$

$$Q(x,y,t) = \begin{cases} Q_{\text{initial}}(x,y) & 0 \le t < \Delta t \\ 0 & t < 0 \end{cases} \tag{7.8}$$

Applying the Fourier-Laplace transforms to equations (7.6) and (7.7) with initial conditions, equation (7.8), and using the shifting theorems for Laplace transform

$$\Delta > 0$$

$$\frac{1}{2\pi}\int_0^{\infty} f(t+\Delta)\, e^{i\omega t}dt = e^{-i\omega\Delta}\tilde{f}(w) - \left(\frac{1}{2\pi}\int_0^{\Delta} f(t)\, e^{i\omega t}dt\right) e^{-i\omega\Delta}$$

$$\frac{1}{2\pi}\int_0^{\infty} f(t-\Delta)\, e^{i\omega t}dt = e^{i\omega\Delta}\tilde{f}(w) + \left(\frac{1}{2\pi}\int_{-\Delta}^{0} f(t)\, e^{i\omega t}dt\right) e^{i\omega\Delta}$$

we have

$$\tilde{K} = -\frac{1}{\Delta x}\left[\sum_{j=-3}^{3} a_j e^{ij\alpha\Delta x}\right]\tilde{E} - \frac{1}{\Delta y}\left[\sum_{j=-3}^{3} a_j e^{ij\beta\Delta y}\right]\tilde{F} + \tilde{H}$$

Define:

$$\overline{\alpha} = \frac{-i}{\Delta x}\sum_{j=-3}^{3} a_j e^{ij\alpha\Delta x}$$

$$\overline{\beta} = \frac{-i}{\Delta y}\sum_{j=-3}^{3} a_j e^{ij\beta\Delta y}$$

We have

$$\boxed{\tilde{K} = -i\overline{\alpha}\tilde{E} - i\overline{\beta}\tilde{F} + \tilde{H}} \tag{7.9}$$

and

$$e^{-i\omega\Delta t}\tilde{Q} - \frac{1}{2\pi}\left(\int_0^{\Delta t} Q_{\text{initial}} e^{i\omega t}dt\right) e^{-i\omega\Delta t}$$

$$= \tilde{Q} + \Delta t\sum_{j=0}^{3} b_j \left(\tilde{K}e^{i\omega j\Delta t} + \frac{1}{2\pi}\underbrace{\left(\int_{-\Delta t}^{0} K(x,y,t)\, e^{i\omega t}dt\right)}_{=0,\ \text{because } K\in[-\Delta t;0]=0} e^{-i\omega\Delta t}\right) \tag{7.10}$$

Equation (7.10) can be rearranged as

$$\begin{aligned}
\left(e^{-i\omega\Delta t}-1\right)\tilde{Q} &= \Delta t\sum_{j=0}^{3} b_j e^{i\omega j\Delta t}\tilde{K} + \frac{e^{-i\omega\Delta t}}{2\pi}\left(\int_0^{\Delta t} Q_{\text{initial}} e^{i\omega t}dt\right)\\
\Rightarrow \left(e^{-i\omega\Delta t}-1\right)\tilde{Q} &= \Delta t\sum_{j=0}^{3} b_j e^{i\omega j\Delta t}\tilde{K} + \frac{e^{-i\omega\Delta t}}{2\pi}\, Q_{\text{initial}} \left.\frac{e^{i\omega t}}{i\omega}\right|_0^{\Delta t}\\
\Rightarrow \left(e^{-i\omega\Delta t}-1\right)\tilde{Q} &= \Delta t\sum_{j=0}^{3} b_j e^{i\omega j\Delta t}\tilde{K} + \frac{Q_{\text{initial}}}{2\pi i\omega}\left(1-e^{-i\omega\Delta t}\right)
\end{aligned}$$

Define $\overline{\omega} = \frac{i\left(e^{-i\omega\Delta t}-1\right)}{\Delta t \sum_{j=0}^{3} b_j e^{i\omega j\Delta t}}$

We have

$$\begin{aligned}
\frac{\overline{\omega}\tilde{Q}}{i} &= \tilde{K} + \frac{Q_{\text{initial}}}{2\pi\omega}\overline{\omega}\\
\Rightarrow & \boxed{-i\overline{\omega}\tilde{Q} = \tilde{K} + \tfrac{Q_{\text{initial}}}{2\pi}\tfrac{\overline{\omega}}{\omega}}
\end{aligned} \tag{7.11}$$

We now have

$$\tilde{K} = -i\overline{\alpha}\tilde{E} - i\overline{\beta}\tilde{F} + \tilde{H} \tag{7.12}$$

$$-\,i\overline{\omega}\tilde{Q} = \tilde{K} + \frac{Q_{\text{initial}}}{2\pi}\frac{\overline{\omega}}{\omega} \tag{7.13}$$

Eliminate $\tilde{K}$

$$\begin{aligned}
\Rightarrow\quad & -i\overline{\omega}\tilde{Q} = -i\overline{\alpha}\tilde{E} - i\overline{\beta}\tilde{F} + \tilde{H} + \frac{Q_{\text{initial}}}{2\pi}\frac{\overline{\omega}}{\omega}\\
\Rightarrow\quad & -i\left(\overline{\omega}\tilde{Q} - \overline{\alpha}\tilde{E} - \overline{\beta}\tilde{F}\right) = \tilde{H} + \frac{Q_{\text{initial}}}{2\pi}\frac{\overline{\omega}}{\omega}\\
\Rightarrow\quad & \left(\overline{\omega}\tilde{Q} - \overline{\alpha}\tilde{E} - \overline{\beta}\tilde{F}\right) = i\left[\tilde{H} + \frac{Q_{\text{initial}}}{2\pi}\frac{\overline{\omega}}{\omega}\right]\\
\Rightarrow\quad & \overline{A}\tilde{Q} = \tilde{T}\tilde{T} = i\left[\tilde{H} + \frac{Q_{\text{initial}}}{2\pi}\frac{\overline{\omega}}{\omega}\right]
\end{aligned}$$

where

$$\overline{A} = \begin{bmatrix}
\overline{\omega}-\overline{\alpha}u_0 & -\rho_0\overline{\alpha} & -\rho_0\overline{\beta} & 0\\
0 & \overline{\omega}-\overline{\alpha}u_0 & 0 & -\overline{\alpha}/\rho_0\\
0 & 0 & \overline{\omega}-\overline{\alpha}u_0 & -\overline{\beta}/\rho_0\\
0 & -\gamma p_0\overline{\alpha} & -\gamma p_0\overline{\beta} & \overline{\omega}-\overline{\alpha}u_0
\end{bmatrix}$$

Replacing α, β, ω in the matrix A (discussed in lecture 6) by $\overline{\alpha}$, $\overline{\beta}$, $\overline{\omega}$, the above analysis shows that the Fourier-Laplace transform of the DRP scheme is the same as the Fourier-Laplace transform of the original partial differential equations. Therefore the two systems must have the same dispersion relations.

7.2 Numerical Stability Requirement (CFL- number)

For entropy and vorticity waves, the dispersion relation is

$$\begin{aligned} \overline{\omega} - \overline{\alpha} u_0 &= 0 \\ \Rightarrow \qquad \overline{\omega}\Delta t &= \overline{\alpha} u_0 \Delta t \end{aligned}$$

From the plot $\overline{\alpha}\Delta x$ vs. $\alpha\Delta x$ (figure on page 32), we can see that for any values of α (or β), the following inequalities hold true

$$\begin{aligned} \overline{\alpha}\Delta x &< 1.7 \qquad \left(\overline{\beta}\Delta y < 1.7\right) \\ \Rightarrow \overline{\omega}\Delta t &\leq \frac{1.7\Delta t}{\Delta x} u_0 \end{aligned}$$

Since the stability analysis for the time discretization shows that $\overline{\omega}\Delta t < 0.4$ is required for stability. This can be shown by a figure $\overline{\omega}\Delta t$ vs. $\omega\Delta t$, similar to $\overline{\alpha}\Delta x$ vs. $\alpha\Delta x$. We then have with the Mach number M and the sound of speed c_0

$$\begin{aligned} \overline{\omega}\Delta t_{max} &= \frac{1.7\Delta t_{max}}{\Delta x} u_0 \\ \Rightarrow \qquad 0.4 &= \frac{1.7\Delta t_{max}}{\Delta x} u_0 \\ \Rightarrow \qquad \Delta t_{max} &= \frac{0.4}{1.7} \cdot \frac{\Delta x}{u_0} = \frac{0.4}{1.7} \cdot \frac{\Delta x}{M \cdot c_0} \\ \Rightarrow \qquad \Delta t &\leq \frac{0.235}{M} \cdot \frac{\Delta x}{c_0} \end{aligned}$$

Thus, the Courant-Friedrichs-Lewy number (CFL) is

$$\frac{\Delta t}{\Delta x} c_0 \leq \frac{0.235}{M}$$

For acoustic waves, the dispersion relation is

$$\overline{\omega} = \overline{\alpha} u_0 + c_0 \sqrt{\overline{\alpha}^2 + \overline{\beta}^2}$$

and we have

$$\begin{aligned} \overline{\alpha}\Delta x, \overline{\beta}\Delta y &< 1.7 \\ \overline{\omega}\Delta t &< 0.4 \end{aligned}$$

The maximum time step is then given by

$$\begin{aligned} \Delta t_{\max} &= \frac{0.4}{1.7\left(M + \sqrt{1 + \left(\frac{\Delta x}{\Delta y}\right)^2}\right)} \frac{\Delta x}{c_0} \\ \Delta t &\leq \frac{0.235}{M + \sqrt{1 + \left(\frac{\Delta x}{\Delta y}\right)^2}} \frac{\Delta x}{c_0} \end{aligned}$$

7.3 Numerical Accuracy Consideration

(a) The group velocity consideration

$$\frac{d\overline{\alpha}}{d\alpha} \leq 1 \pm 0.003 \Rightarrow \overline{\alpha}\Delta x < 0.9$$

(reasonably accurate prediction for group velocity with an error of ±0.3 %)

(b) Consideration of the numerical damping in the time discretization

$$\overline{\omega}\Delta t \leq 0.19$$

$$\boxed{\Delta t \leq \frac{0.211}{M+\sqrt{1+\left(\frac{\Delta x}{\Delta y}\right)^2}} \cdot \frac{\Delta x}{c_0}}$$

based on accuracy consideration

and

$$\boxed{\Delta t \leq \frac{0.235}{M+\sqrt{1+\left(\frac{\Delta x}{\Delta y}\right)^2}} \cdot \frac{\Delta x}{c_0}}$$

based on stability consideration

The requirement for numerical accuracy is slightly more stringent than that for numerical stability.

7.4 Group Velocity

The dispersion relation in general is $\omega = \omega(\alpha, \beta)$. The group velocity is therefore expressed as

$$\vec{v}_{\text{gr}} = \frac{\partial \omega}{\partial \alpha}\hat{e}_x + \frac{\partial \omega}{\partial \beta}\hat{e}_y$$

For the entropy and the vorticity waves, the dispersion relation is $\omega = u_0\alpha$.

$$\Rightarrow \frac{\partial \omega}{\partial \alpha} = u_0 \qquad \frac{\partial \omega}{\partial \beta} = 0.$$

Hence

$$\vec{v}_{\text{gr}} = u_0\hat{e}_x$$

The wave is convected downstream at the speed of the mean flow.

For the acoustic waves, the dispersion relations

$$\omega = \alpha u_0 \pm c_0\sqrt{\alpha^2+\beta^2}$$

$$\vec{v}_{\text{gr}} = \left[u_0 \pm \frac{\alpha c_0}{\sqrt{\alpha^2+\beta^2}}\right]\hat{e}_x \pm \frac{\beta c_0}{\sqrt{\alpha^2+\beta^2}}\hat{e}_y$$

For waves propagating in the x-direction ($\beta = 0$), we have

$$\vec{v}_{\text{gr}} = (u_0 \pm c_0)\hat{e}_x$$

7.5 Group Velocity for DRP Schemes

$$\begin{aligned}
\vec{v}_{\text{gr,DRP}} &= \frac{\partial \omega}{\partial \alpha}\hat{e}_x + \frac{\partial \omega}{\partial \alpha}\hat{e}_y \\
&= \frac{\partial \omega}{\partial \overline{\omega}}\frac{\partial \overline{\omega}}{\partial \overline{\alpha}}\frac{\partial \overline{\alpha}}{\partial \alpha}\hat{e}_x + \frac{\partial \omega}{\partial \overline{\omega}}\frac{\partial \overline{\omega}}{\partial \overline{\beta}}\frac{\partial \overline{\beta}}{\partial \beta}\hat{e}_y \\
&= \frac{\frac{\partial \overline{\omega}}{\partial \overline{\alpha}}\frac{d\overline{\alpha}}{d\alpha}}{\frac{d\overline{\omega}}{d\omega}}\hat{e}_x + \frac{\frac{\partial \overline{\omega}}{\partial \overline{\beta}}\frac{d\overline{\beta}}{d\beta}}{\frac{d\overline{\omega}}{d\omega}}\hat{e}_y
\end{aligned}$$

The dispersion relation for acoustic waves is

$$\overline{\omega} = u_0\overline{\alpha} \pm c_0\sqrt{\overline{\alpha}^2 + \overline{\beta}^2}$$

$$\Rightarrow \vec{v}_{\text{gr,DRP}} = \frac{\left[u_0 \pm \frac{c_0\overline{\alpha}}{\sqrt{\overline{\alpha}^2+\overline{\beta}^2}}\right]\frac{d\overline{\alpha}}{d\alpha}}{\frac{d\overline{\omega}}{d\omega}}\hat{e}_x + \frac{\left[\pm\frac{c_0\overline{\beta}}{\sqrt{\overline{\alpha}^2+\overline{\beta}^2}}\right]\frac{d\overline{\beta}}{d\beta}}{\frac{d\overline{\omega}}{d\omega}}\hat{e}_y$$

For waves propagating in the x-direction ($\beta = \overline{\beta} = 0$)

$$\vec{v}_{\text{gr,DRP}} = \frac{(u_0 \pm c_0)\frac{d\overline{\alpha}}{d\alpha}}{\frac{d\overline{\omega}}{d\omega}}\hat{e}_x$$

Compared to the analytical group velocity, an error is introduced by the DRP with $\frac{d\overline{\alpha}}{d\alpha}$ and $\frac{d\overline{\omega}}{d\omega}$. As seen before, the requirements on the numerical method for correct predictions are

$$\begin{aligned}
\frac{d\overline{\alpha}}{d\alpha} &= \frac{d(\overline{\alpha}\Delta x)}{d(\alpha\Delta x)} = 1 \\
\frac{d\overline{\omega}}{d\omega} &= \frac{d(\overline{\omega}\Delta t)}{d(\omega\Delta t)} = 1
\end{aligned}$$

For the entropy and the vorticity waves, the dispersion relation is

$$\overline{\omega} = u_0\overline{\alpha}$$

$$\Rightarrow \vec{v}_{\text{gr,DRP}} = \frac{u_0\frac{d\overline{\alpha}}{d\alpha}}{\frac{d\overline{\omega}}{d\omega}}\hat{e}_x$$

8 The Short Wave Component of Finite Difference Schemes

Consider the initial value problem associated with the linearized Euler equations in one dimension without mean flow.

The dimensionless linearized momentum equations is obtained by using normalized variables:

$$u' = U \cdot c_\infty \qquad x = X \cdot R$$
$$p' = P \cdot \rho_0 \cdot c_\infty^2 \qquad \rho_0 = \varrho_0 \cdot \rho_\infty$$
$$t = T\frac{R}{c_\infty^2} \qquad \text{with } R \text{ being a specific length}$$

Applied to the linarized momentum equation

$$\rho_0 \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0$$

we have

$$\varrho_0 \frac{\partial U}{\partial T} \frac{\rho_\infty c_\infty^2}{R} + \frac{\partial P}{\partial X} \frac{\rho_0 c_\infty^2}{R} = 0$$

and with $\varrho_0 = 1$ or $\rho_0 = \rho_\infty$ we have the dimensionless linearized momentum equation

$$\frac{\partial U}{\partial T} + \frac{\partial P}{\partial X} = 0 \tag{8.1}$$

In the same manner we get the dimensionless linearized energy equation

$$\frac{\partial P}{\partial T} + \frac{\partial U}{\partial X} = 0 \tag{8.2}$$

With the initial conditions

$$U(t = 0, x) = 0 \qquad P(t = 0, x) = f(x)$$

the exact solution is a plane wave

$$P(x, t) = \frac{1}{2}\left[f(x - t) + f(x + t)\right]$$

Consider the Gaussian function

$$f(x) = e^{-ax^2}$$

with its Fourier transform

$$\widetilde{f}(\alpha) = \frac{1}{\sqrt{2a}} e^{-\frac{\alpha^2}{4a}}$$

The half width of the Gaussian function $f(x)$ is

$$\sqrt{\frac{\ln 2}{a}} = \omega$$

$$a = \frac{\ln 2}{d} \qquad d = 25, 9, 4, 1, 0.04 \text{ (eg.)}$$

On discretizing (8.1) and (8.2) using the 7-point-stencil DRP scheme with $\Delta x = 1$

$$E_l^{(n)} = -\sum_{j=-3}^{3} a_j P_{l+j}^{(n)}$$

$$F_l^{(n)} = -\sum_{j=-3}^{3} a_j U_{l+j}^{(n)}$$

$$U_l^{(n+1)} = U_l^{(n)} + \Delta t \sum_{j=-3}^{3} b_j E_l^{(n-j)}$$

$$P_l^{(n+1)} = P_l^{(n)} + \Delta t \sum_{j=0}^{3} b_j F_l^{(n-j)}$$

The initial conditions are

$$U_l^{(n)} = 0 \qquad n \leq 0$$

$$P_l^{(n)} = \begin{cases} e^{-ax_l^2} & n = 0 \\ 0 & n < 0 \end{cases}$$

The numerical solutions for $d = 25.9$ are given at $t = 200\Delta t$, $2000\Delta t$ and $4000\Delta t$ (with $\Delta t = 0.1$) and for $d = 4$, 1 and 0.04 at $t = 200\Delta t$. As we can see the numerical solutions deviated as the value of ω decreases. If we take a look at the plot $\frac{d\overline{\alpha}}{d\alpha}$ vs. $\alpha\Delta x$ the group velocity for the short wave is less than 1. The high wavenumber components have negative group velocity. We define dispersive waves and parasite waves to distinguish the short waves with positive and negative group velocity.

$$\frac{d\overline{\alpha}}{d\alpha} = \begin{cases} 1.0 & \alpha \leq 1.2 \\ \text{dispersive waves} & 1.2 \leq \alpha \leq 2.0 \\ \text{parasite waves} & 2.0 \leq \alpha \leq \pi \end{cases}$$

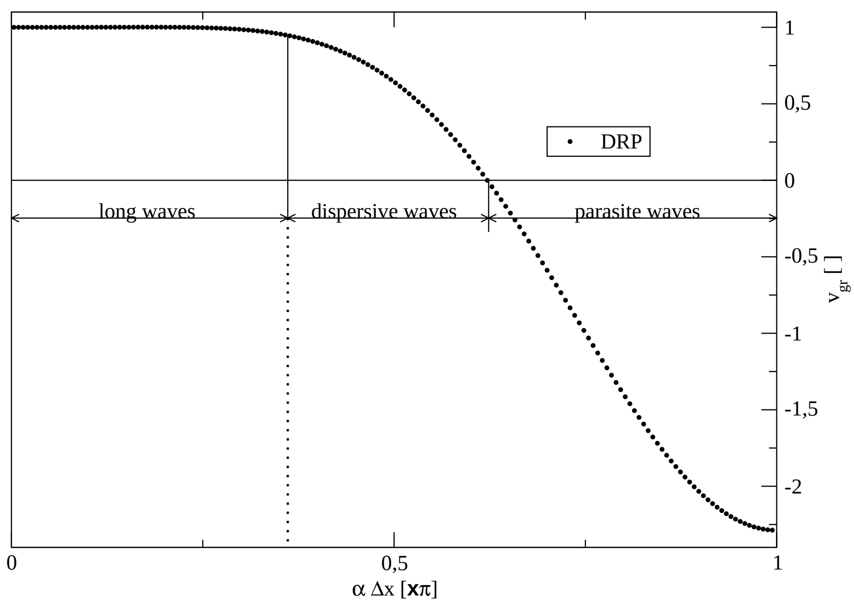


Figure 8.1: Group velocity of the DRP

In order to study the wave propagation characteristics of the short waves of the finite difference schemes, we consider a discontinuous initial condition (rectangle function),

$$f(x) = H(x+M) - H(x-M)$$

where M is a large positive number and $H(x)$ is the unit step function or Heaviside function.

$$H(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & x > 0 \end{cases}$$

The exact solution of such artificial condition is

$$P(x,t) = \frac{1}{2}[H(x-t+M) - H(x-t-M)] + \frac{1}{2}[H(x+t+M) - H(x+t-M)]$$

The solution by the numerical scheme (DRP) in physical space and time can be found by inverting the Fourier-Laplace transform.

Step 1 Write finite difference equations with continuous variables x and t

$$\begin{aligned} U(x, t+\Delta t) &= U(x,t) + \Delta t \sum_{j=-3}^{3} b_j E(x, t - j\Delta t) \\ P(x, t+\Delta t) &= P(x,t) + \Delta t \sum_{j=-3}^{3} b_j F(x, t - j\Delta t) \\ E(x,t) &= -\sum_{j=-3}^{3} a_j P(x + j\Delta x, t) \\ F(x,t) &= -\sum_{j=-3}^{3} a_j U(x + j\Delta x, t) \end{aligned}$$

The initial conditions:

$$\begin{aligned} U &= \quad 0 \qquad\qquad\qquad\qquad\qquad t < \Delta t \\ P &= \begin{cases} H(x+M) - H(x-M) & 0 \le t \le \Delta t \\ 0 & t < 0 \end{cases} \end{aligned}$$

Step 2 Solve the above initial value problem by the Fourier- Laplace transform.

Step 3 The Fourier-Laplace transform of P is given by

$$\widetilde{p} = \frac{i}{2\pi} \left(\frac{\overline{\omega}^2}{\omega} \right) \frac{\widetilde{f}}{\overline{\omega}^2 - \overline{\alpha}^2}$$

where

$$\begin{aligned} \widetilde{f} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} [H(x+M) - H(x-M)]\, e^{-i\alpha x} dx \\ \widetilde{f} &= \frac{\sin \alpha\ M}{\pi \alpha} \end{aligned}$$

Step 4 The solution in physical space and time

$$P(x,t) = \frac{i}{2\pi}\int_\Gamma \int_{-\infty}^{\infty} \left(\frac{\overline{\omega}^2}{\omega}\right) \frac{\widetilde{f}}{\overline{\omega}^2 - \overline{\alpha}^2}\ e^{i(\alpha x - \omega t)}\ d\alpha d\omega$$

There are two poles at $\overline{\omega}(\omega) = \pm\overline{\alpha}(\alpha)$. By the Residual Theorem ($t \to \infty$), we have

$$\begin{aligned} P(x,t) =& \frac{1}{2}\int_{-\infty}^{\infty} \widetilde{f}(\alpha)\ e^{i(\alpha x - \overline{\alpha} t)} d\alpha \\ &+ \frac{1}{2}\int_{-\infty}^{\infty} \widetilde{f}(\alpha)\ e^{i(\alpha x + \overline{\alpha} t)} d\alpha \end{aligned}$$

Step 5 Evaluation of the solution $P(x,t)$. Divide the integral (the first integral) into four seperate integrals.

$$\begin{aligned} P(x,t) =& \frac{1}{2}\left(\int_{-\infty}^{-\pi} + \int_{\pi}^{\infty} \widetilde{f}\ e^{i(\alpha x - \overline{\alpha} t)} d\alpha\right) && I_1 \\ &+ \frac{1}{2}\left(\int_{-\pi}^{-2.0} + \int_{2.0}^{\pi} \widetilde{f}\ e^{i(\alpha x - \overline{\alpha} t)} d\alpha\right) && I_2 \\ &+ \frac{1}{2}\left(\int_{-2.0}^{-1.2} + \int_{1.2}^{2.0} \widetilde{f}\ e^{i(\alpha x - \overline{\alpha} t)} d\alpha\right) && I_3 \\ &+ \frac{1}{2}\left(\int_{-1.2}^{1.2} \widetilde{f}\ e^{i(\alpha x - \overline{\alpha} t)} d\alpha\right) && I_4 \end{aligned}$$

The above four integrals are referred as I_1, I_2, I_3 and I_4.
I_1: contribution from ultra-short waves. ($\lambda < 2\Delta x$)
I_2: contribution from parasite waves
I_3: contribution from dispersive waves
I_4: contribution from long waves

In order to evaluate these integrals, we need to know $\overline{\alpha}$ as a function of α. To simplify the analysis, the graph of the $\frac{d\overline{\alpha}}{d\alpha}$ curve is approximated by the analytical formula

$$\frac{d\overline{\alpha}}{d\alpha} = \begin{cases} 1.0 & \text{(long waves)} & \alpha \leq 1.2 \\ 1 - \frac{(\alpha-1.2)^2}{0.64} & \text{(dispersive waves)} & 1.2 \leq \alpha \leq 2.0 \\ -2.75(\alpha - 2.0) & \text{(parasite waves)} & 2.0 \leq \alpha \leq \pi \end{cases}$$

I_1: ($\lambda < 2\Delta x$) It is not important to the present discretized solution because the ultra-short waves are not resolved. Neglected.

$$\begin{aligned} I_4 &= \frac{1}{2\pi}\int_{-\infty}^{\infty} [H(y+M) - H(y-M)]\ \frac{\sin[1.2(x-t-y)]}{x-y-t}\ dy \\ I_4 &= \frac{1}{2\pi}\{S_i[1.2(x-t+M)] - S_i[1.2(x-t-M)]\} \end{aligned}$$

where

$$S_i(x) = \int_0^x \frac{\sin y}{y}\ dy$$

I_2 and I_3 can be evaluated for large time by the method of stationary phase.

The method of stationary phase provides the following formula.

$$\lim_{t\to\infty}\int g(\beta)e^{ih(\beta)t}d\beta \approx \sqrt{\frac{2\pi}{t\,|h''(\beta_s)|}}g(\beta_s)e^{ih(\beta_s)t+i\frac{\pi}{4}\operatorname{sgn}(h''(\beta_s))}$$

where

$$\left.\frac{dh(\beta)}{d\beta}\right|_{\beta=\beta_s} = 0$$

β_s is called the stationary point of the phase function h and sgn the sign of.

I_2 and I_3 can be written as

$$\int \widetilde{f}(\alpha)e^{i[\alpha\frac{x}{t}-\overline{\alpha}(\alpha)]t}\,d\alpha$$

For large t, the phase function is $h = \alpha\frac{x}{t} - \overline{\alpha}(\alpha)$.

The stationary point is given by

$$\begin{aligned} \frac{dh}{d\alpha} &= 0 \\ \rightarrow \frac{x}{t} &= \frac{d\overline{\alpha}}{d\alpha} \\ \rightarrow x &= \frac{d\overline{\alpha}}{d\alpha}t \end{aligned}$$

$\frac{d\overline{\alpha}}{d\alpha}$ is the effective speed of propagation for the wave with wavenumber α.

Parasite waves

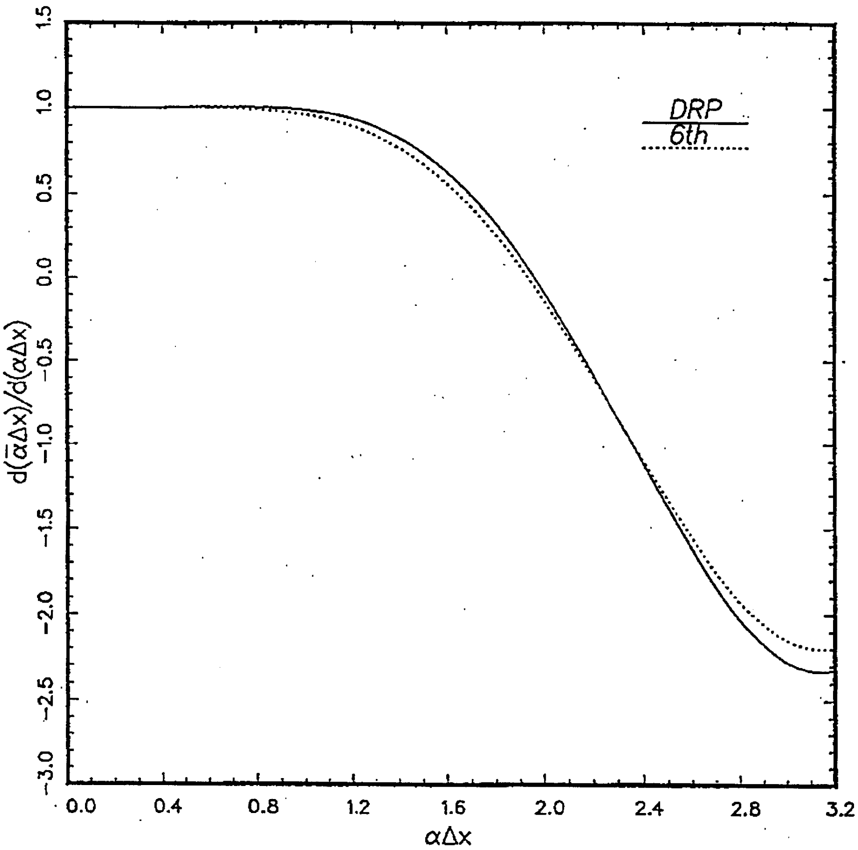
$$I_2 \approx \sqrt{\frac{0.727}{\lambda t}}\frac{\sin\left[\left(2.0-\frac{x}{2.75t}\right)M\right]}{2.0-\frac{x}{2.75t}}\cos\left[\left(2x-1.733t-0.1818\frac{x^2}{t}\right)+\frac{\pi}{4}\right]$$

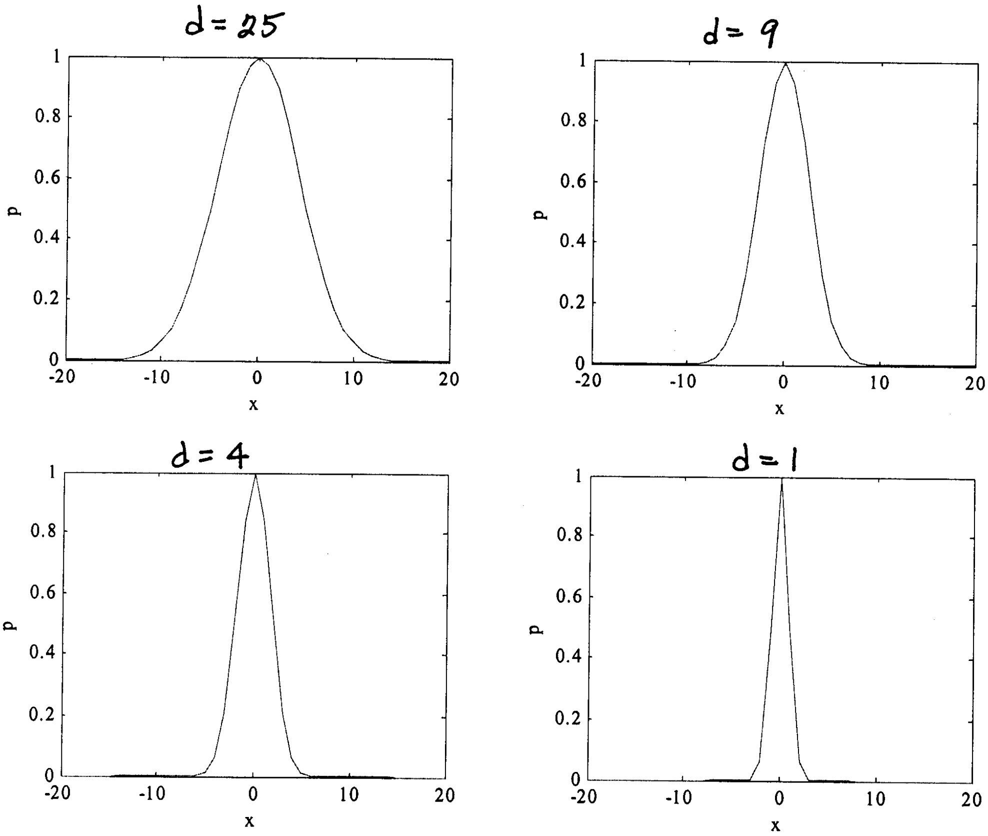
$$-3.139t < x < 0$$

Dispersive waves

$$I_3 \approx \sqrt{\frac{0.8}{\lambda t}}\frac{1}{\left(1-\frac{x}{t}\right)^{\frac{1}{4}}}\ \frac{\sin\left[1.2+0.8\sqrt{1-\frac{x}{t}}M\right]}{1.2+0.8\sqrt{1-\frac{x}{t}}}\ \cos\left[(x-t)\left(1.2+0.533\sqrt{1-\frac{x}{t}}\right)+\frac{\pi}{4}\right]$$

$$0 < x < t$$



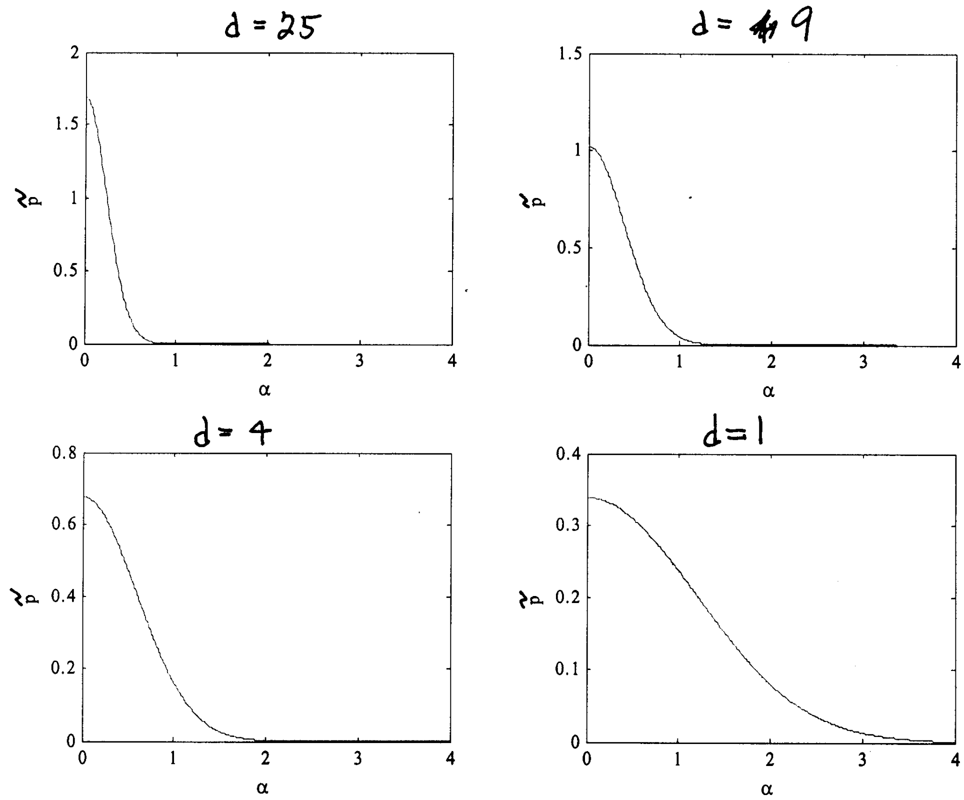


Initial condition

$p(x) = e^{-ax^2}$

where

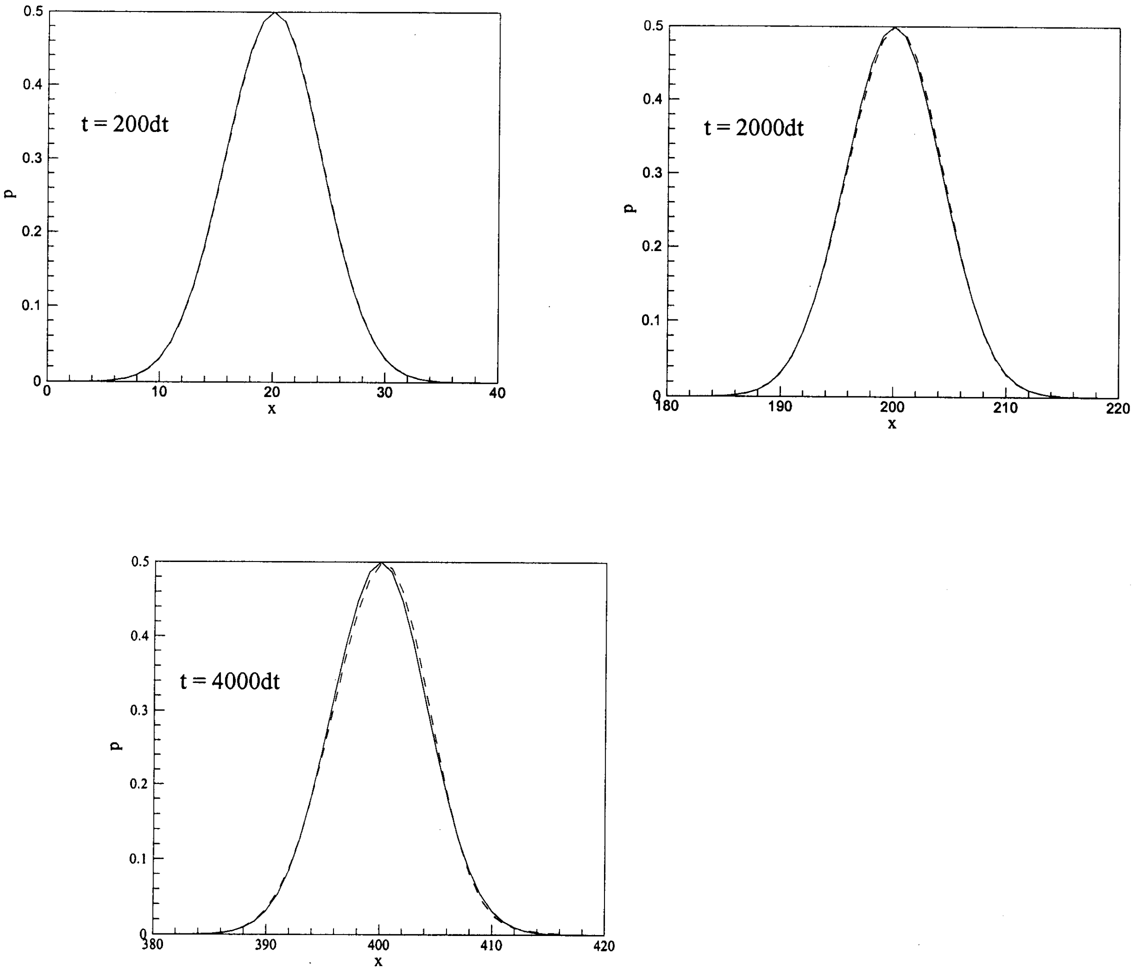
$a = \ln 2 / d$



The Fourier transform of $p(x)$

$$\tilde{p}(\alpha) = \frac{1}{2\sqrt{\pi a}} e^{-\alpha^2/4a}$$

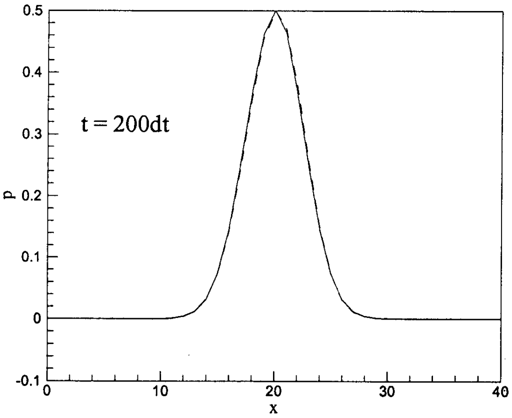
$$a = \ln 2/d$$

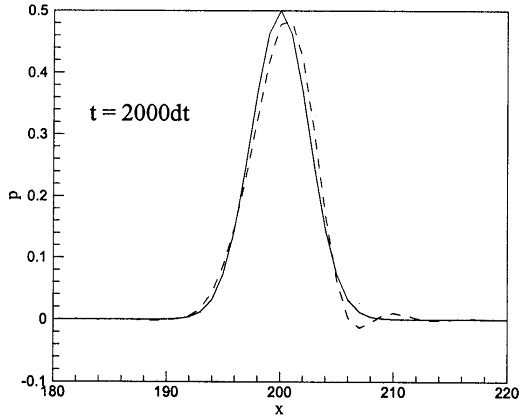


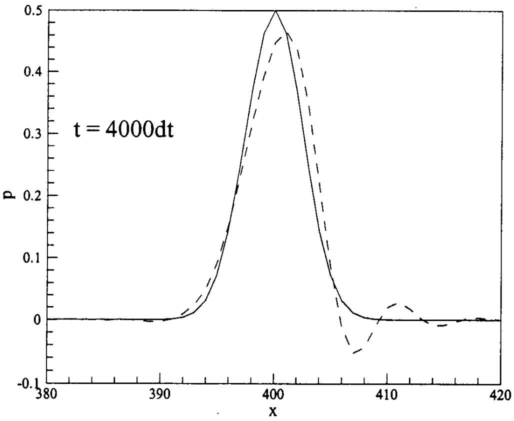
d = 25

3

d= 9

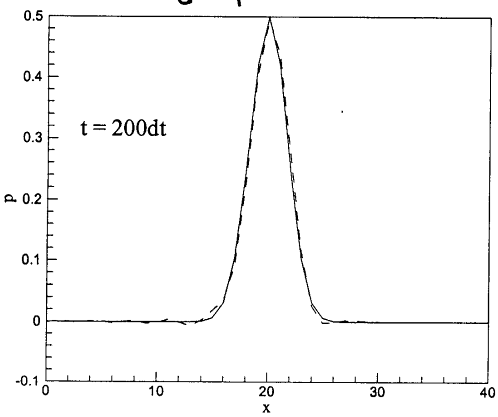




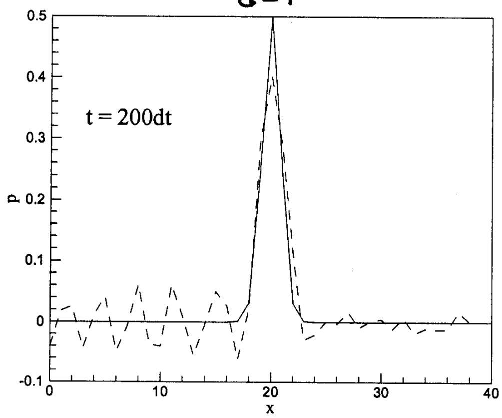


4

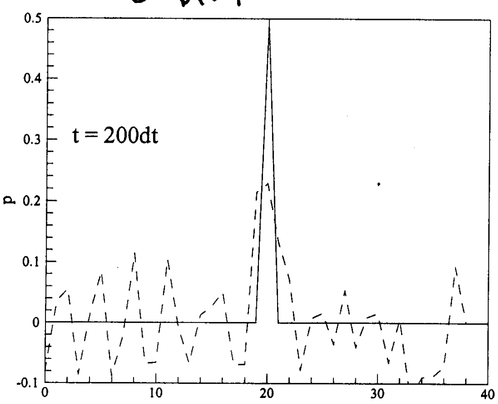
d= 4

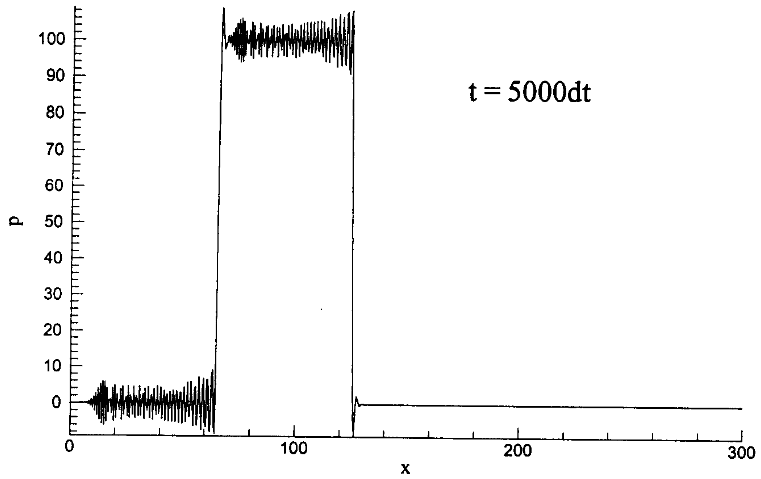


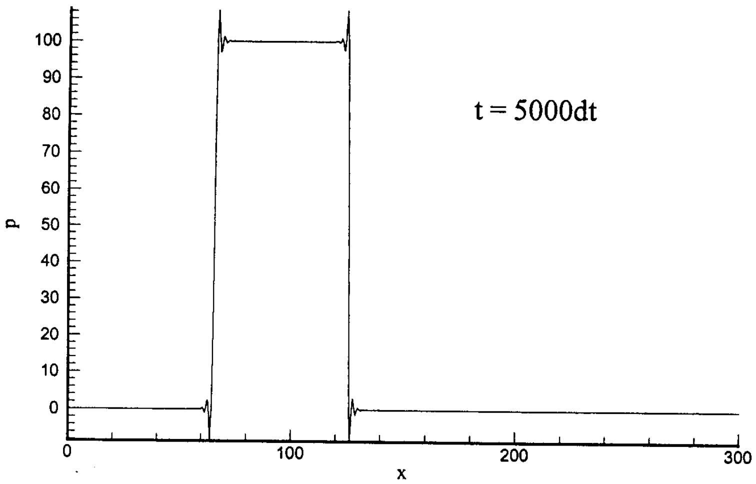
d=1

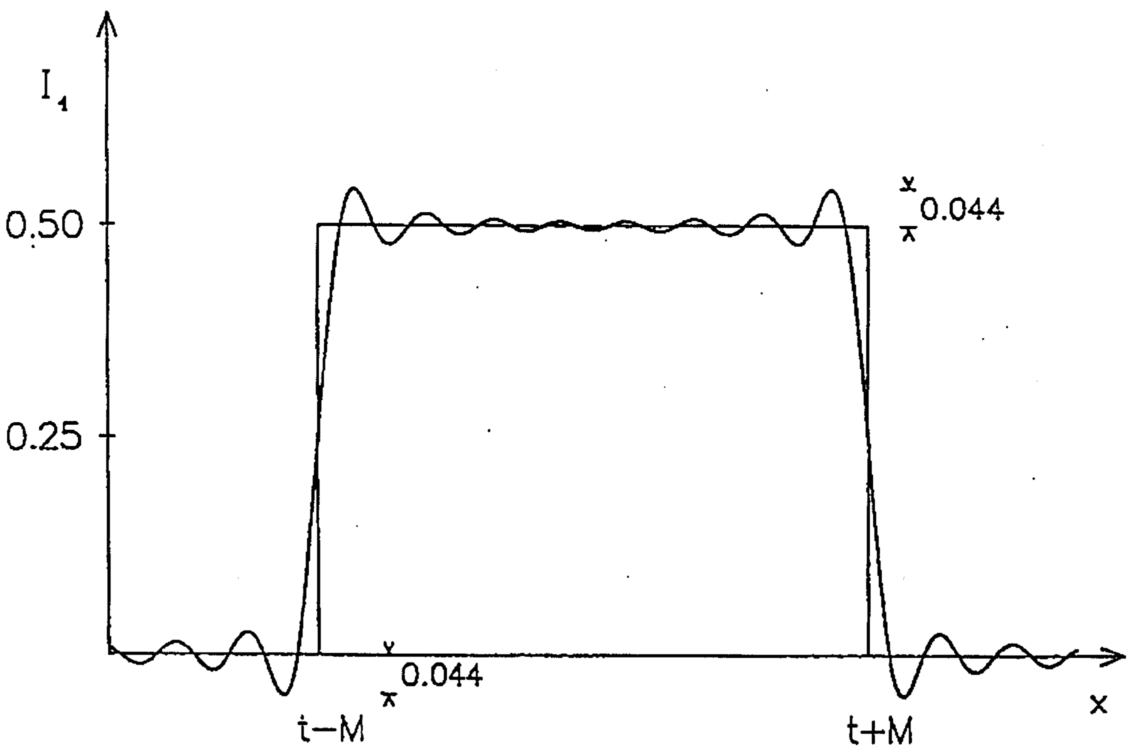


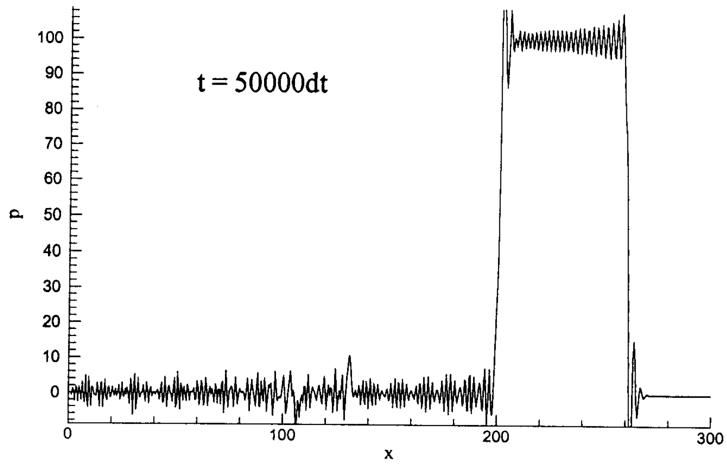
d=0.04

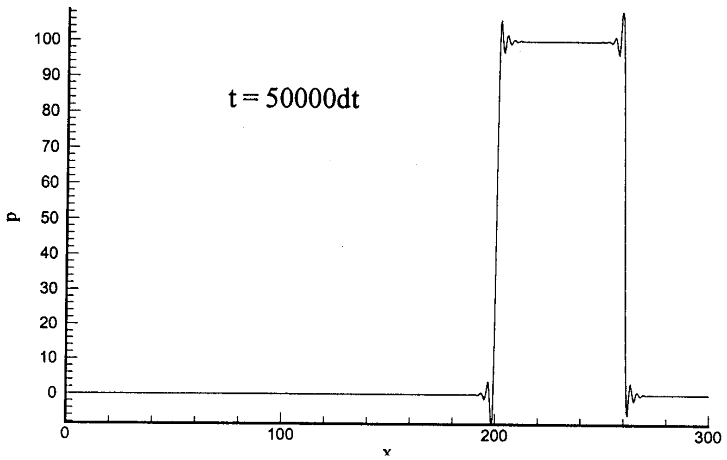


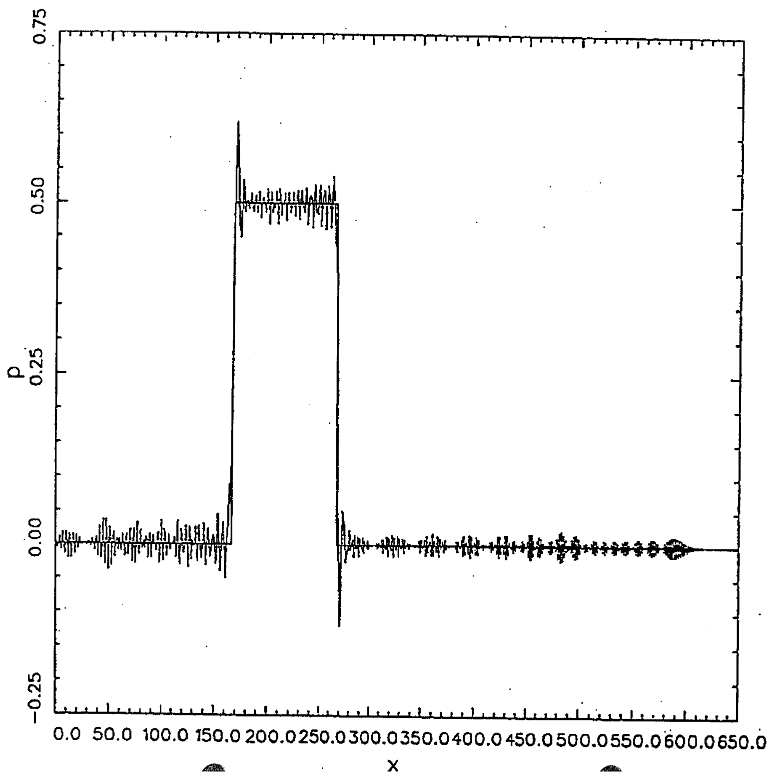

t = 5000dt
p
x


t = 5000dt
p
x


I
0.50
0.25
0.044
0.044
t−M
t+M
x


t = 50000dt
p
x
0
10
20
30
40
50
60
70
80
90
100
0
100
200
300


t = 50000dt
p
0
10
20
30
40
50
60
70
80
90
100
0
100
200
300


p
x
−0.25
0.00
0.25
0.50
0.75
0.0 50.0 100.0 150.0 200.0 250.0 300.0 350.0 400.0 450.0 500.0 550.0 600.0 650.0

9 Selective Artificial Damping (SAD)

Purpose of artificial damping:	to remove numerical contaminants of a computed solution.
Important points:	1) Selectively damp out the short waves 2) have minimal effect on the long waves

9.1 Basic concept

$$\frac{\partial u'}{\partial t} + \frac{1}{\rho_0}\frac{\partial p'}{\partial x} = D(x)$$

$$\uparrow$$

Dimensional momentum eq. (no mean flow) of the linearized Euler eqs.
Discretization using 7-point-stencil,

$$\frac{\partial u_l}{\partial t} + \frac{1}{\rho_0}\frac{1}{\Delta x}\sum_{j=-3}^{3} a_j p_{l+j} = D_l$$

9.2 Assumption

D_l is proportional to the value of u_l within the stencil.

$$\frac{du_l}{dt} + \frac{1}{\rho_0}\frac{1}{\Delta x}\sum_{j=-3}^{3} a_j p_{l+j} = -\frac{v}{(\Delta x)^2}\sum_{j=-3}^{3} d_j u_{l+j} \tag{9.1}$$

Where d_j are the weight coefficients, v is the artificial kinetic viscosity.

$$\left[\frac{v}{(\Delta x)^2}\right] = \left[\frac{1}{\text{time}}\right]$$

d_j's are pure numbers.

9.3 Approach

Choose d_j so that the artificial damping would be effective mainly for high wavenumber or short waves. The Fourier transform of the continuous form of eq. (9.1)

$$\frac{d\tilde{u}}{dt} + \ldots = -\frac{v}{(\Delta x)^2}\sum_{j=-3}^{3} d_j e^{ij\alpha\Delta x}\,\tilde{u}$$

$$\uparrow$$

If neglecting the terms not shown here, the solution is

$$\tilde{u} \sim e^{-i\frac{v}{(\Delta x)^2}\tilde{D}(\alpha\Delta x)\,t}$$

where

$$\tilde{D}(\alpha\Delta x) = \sum_{j=-3}^{3} d_j e^{ij\alpha\Delta x}$$

Three conditions on $\tilde{D}(\alpha\Delta x)$.

(a) $\tilde{D}(\alpha\Delta x)$ should be a positive even function of $\alpha\Delta x$.

$$d_j = d_{-j}$$
$$\rightarrow \tilde{D}(\alpha\Delta x) = d_0 + 2\sum_{j-1}^{3} d_j \cos\left[j\alpha\Delta x\right]$$

(b) There should be no damping for long waves.

$$\tilde{D}(\alpha\Delta x) \rightarrow 0 \text{ as } \alpha\Delta x \rightarrow 0$$

This requires

$$d_0 + 2\sum_{j-1}^{3} d_j = 0 \tag{9.2}$$

(c) For convenience, $\tilde{D}(\alpha\Delta x)$ is normalized so that

$$\tilde{D}(\pi) = 1 \tag{9.3}$$

$$\tilde{D}(\alpha\Delta x) = d_0 + 2\left[d_1\cos\alpha\Delta x + d_2\cos 2\alpha\Delta x + d_3\cos 3\alpha\Delta x\right]$$

We would like to have the properties:

$\tilde{D}(\alpha\Delta x)$ small for small $\alpha\Delta x$ but large $\tilde{D}(\alpha\Delta x)$ when $\alpha\Delta x \rightarrow \pi$

Take a Gaussian function centered at π with half-width σ

$$f(\alpha\Delta x) = e^{-\ln 2\left(\frac{\alpha\Delta x-\pi}{\sigma}\right)^2}$$

when $\alpha\Delta x = \pi$: $f(\alpha\Delta x) = 1$
when $\alpha\Delta x = 0$ (or small): $f(\alpha\Delta x) \rightarrow 0$ (or small)

$\rightarrow$ then the weight coefficients d_j are determined such that the integral

$$\int_0^\beta \left[\tilde{D}(\alpha\Delta x) - e^{-\ln 2\left(\frac{\alpha\Delta x-\pi}{\sigma}\right)^2}\right]^2 d(\alpha\Delta x)$$

is a minimum.

With eqs. (9.2), (9.3) and the above minimization condition, we can determine the coefficients d_j. β is a parameter that can be adjusted to yield the most desirable properties for $\tilde{D}$.

For the discontinuous solution (boxcar problem).

$$\sigma = 0.3\pi \qquad \beta = 0.65\pi$$

$$\begin{aligned} d_0 &= 0.3276986608 \\ d_1 = d_{-1} &= -0.235718815 \\ d_2 = d_{-2} &= 0.0891506696 \\ d_3 = d_{-3} &= -0.0142811847 \end{aligned}$$

9.4 Numerical Implementation

$$\left.\begin{array}{l}\frac{\partial U}{\partial T}-\frac{\partial P}{\partial X}=0\\ \frac{\partial P}{\partial T}+\frac{\partial U}{\partial X}=0\end{array}\right\}\quad \begin{array}{c}\text{1-d dimensionless Euler eq.}\\ \text{(linearized, no mean flow)}\end{array}$$

$$P=\begin{cases} H(x+m)-H(x-m) & t=0\\ 0 & t<0\end{cases}$$

$$U=0 \qquad\qquad t=0$$

$$\begin{aligned}
U_l^{(n+1)} &= U_l^{(n)}+\Delta t\sum_{j=0}^{3} b_j E_l^{(n-j)}\\
P_l^{(n+1)} &= P_l^{(n)}+\Delta t\sum_{j=0}^{3} b_j F_l^{(n-j)}\\
E_l^{(n)} &= -\sum_{j=-3}^{3} a_j P_{l+j}^{(n)}-\frac{1}{R}\sum_{j=-3}^{3} d_j U_{l+j}^{(n)}\\
F_l^{(n)} &= -\sum_{j=-3}^{3} a_j U_{l+j}^{(n)}-\frac{1}{R}\sum_{j=-3}^{3} d_j P_{l+j}^{(n)}
\end{aligned}$$

$$U_l^{(n)}=0 \qquad (n\le 0)$$

$$P_l^{(n)}=\begin{cases} H(x_l+M)-H(x_l-M) & n=0\\ 0 & n<0\end{cases}$$

where $R=\frac{c_0\Delta x}{v}$ is the artificial mesh Reynolds number.
$\frac{1}{R}=0.3 \;\Rightarrow$ compare the Results.

Background damping:

(a) +0 avoid the generation of parasite waves through discontinuous initial or boundary conditions

(b) Nonlinearities

(c) Rapid changes at boundary interfaces

Some useful artificial selective damping stencils:

7-point damping stencil ($\sigma = 0.2\pi$)
$d_0 = 0.2873928425$
$d_1 = d_{-1} = -0.2261469518$
$d_2 = d_{-2} = 0.106305788$
$d_3 = d_{-3} = -0.0238530482$

5-point damping stencil (Taylor, not optimized)
$d_0 = 0.375$
$d_1 = d_{-1} = -0.25$
$d_2 = d_{-2} = 0.0625$

3-point damping stencil (not optimized)
$d_0 = 0.5$
$d_1 = d_{-1} = -0.25$

9.5 Excessive Damping

(a) Excessive damping can cause "artificial viscous diffusion".

$$\boxed{\frac{1}{R} = 5}$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$
$$u(x,0) = 0,5e^{-\ln 2\left(\frac{x}{5}\right)^2}$$

The Fourier transform of a Gaussian function is also a Gaussian function. Artificial damping reduces the amplitude of the pulse in the wavenumber space unevenly:

(a) No reduction at zero wave number.

(b) The reduction increases with α increases.

Therefore the pulse becomes narrower in the wavenumber space. This results the physical waveform spreads out in the physical space.

(b) Excessive damping can cause "numerical instability".

$$\boxed{\frac{1}{R} = 15}$$

Why?

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad -\infty < x < \infty$$
$$u(x,0) = 0,5e^{-\ln 2\left(\frac{x}{5}\right)^2}$$

The continuous form of the discretized eqs.

$$u\ (x, t+\Delta t) = u\ (x,t) + \Delta t \sum_{j-0}^{3} b_j \kappa\ (x, t - j\Delta t)$$
$$\kappa\ (x,t) = -\tfrac{1}{\Delta x} \sum_{j=-3}^{3} a_j u\ (x + j\Delta x, t)$$
$$-\tfrac{1}{R} \sum_{j=-3}^{3} d_j u\ (x + j\Delta x, t)$$

Let us take the Fourier-Laplace transform of the above equations.

$$e^{-iw\Delta t}\tilde{u} = \tilde{u} + \Delta t \sum_{j-0}^{3} b_j e^{-iwj\Delta t}\tilde{k}$$
$$\tilde{k} = -\tfrac{1}{\Delta x} \sum_{j=-3}^{3} a_j e^{-i\alpha j\Delta x}\tilde{u} - \tfrac{1}{R} \sum_{j=-3}^{3} d_j e^{-i\alpha j \Delta x}\bar{u}$$

Eliminate $\tilde{k}$ and use

$$\bar{w} = \frac{i\left(e^{-iw\Delta t} - 1\right)}{\Delta t \sum_{j=0}^{3} b_j e^{-iw\Delta t}}$$
$$\bar{\alpha}_{(\alpha\Delta x)} = \tfrac{-i}{\Delta x} \sum_{j=-3}^{3} a_j e^{ij\alpha\Delta x}$$
$$\bar{D}_{(\alpha\Delta x)} = \sum_{j=-3}^{3} d_j e^{ij\alpha\Delta x}$$

We have

$$-i\bar{w}\tilde{u} + i\bar{\alpha}\tilde{u} = -\frac{1}{R}\tilde{D}\ (\alpha\Delta x)\ \tilde{u}$$

This the dispersion relation is

$$\bar{w}u = \bar{\alpha} - \frac{i}{R}\tilde{D}\ (\alpha\Delta x)\ u$$

Consider the wave with $\alpha\Delta x = \pi\ (\lambda = 2\Delta x)$

$$\begin{pmatrix} \frac{\pi}{\alpha} = \Delta x \\ \pi = \alpha\Delta x \end{pmatrix}$$

We have noted that

$$\begin{array}{c} \bar{\alpha}\ (\pi) = 0.0 \\ \bar{D}\ (\pi) = 1.0 \end{array}$$

$$\bar{w} = -\frac{1}{R} \Rightarrow \bar{w}\Delta t = -\frac{i\Delta t}{R}$$

When $\bar{w}\Delta t$ is a complex number, the time discretization scheme (optimized multi-time level scheme) is stable

$$\begin{array}{l} \text{if } I_m\,(\bar{w}\Delta t) > -0.29 \\ \quad \Rightarrow \frac{\Delta t}{R} < 0.29 \qquad \text{stable} \end{array}$$

In the numerical example shown, $\Delta t = 0.02$ and $\frac{1}{R} = 0.15$ so that $\frac{\Delta t}{R} = 0.3$. We see the numerical instability!

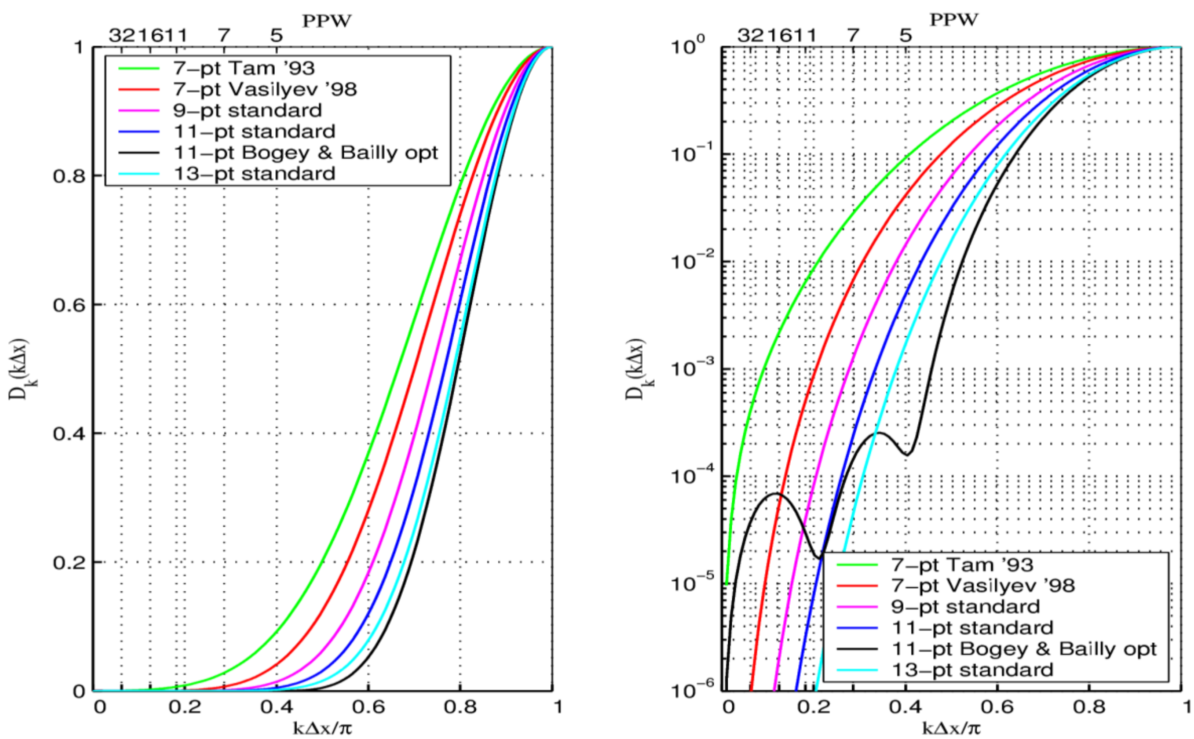


Figure 9.1: Filtering characteristics of different filter stencils

10 Filtering approaches for the DRP scheme

10.1 Problem Statement – Why SAD makes us sad

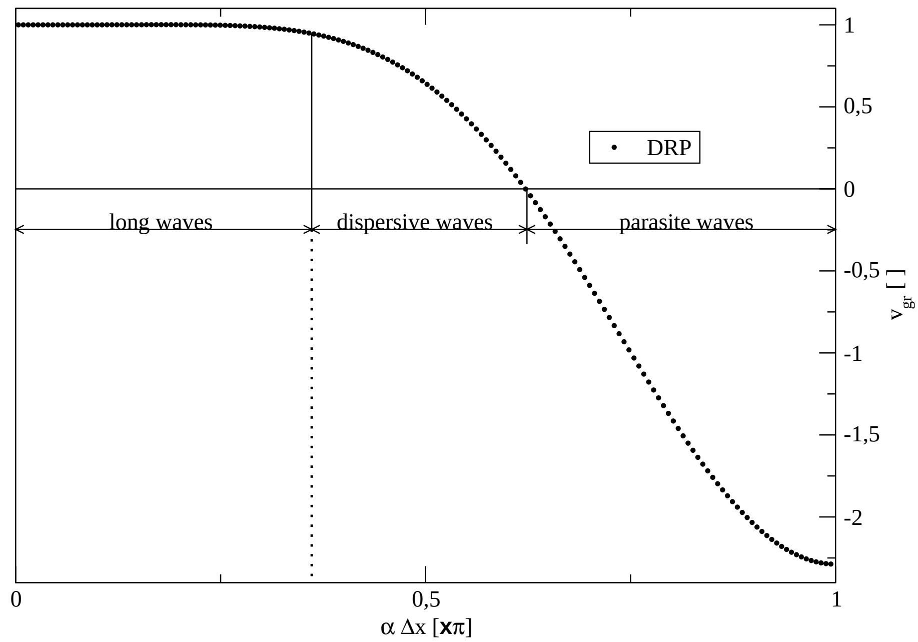


Figure 10.1: Group velocity of the DRP

As seen in figure 10.1 the DRP has a usable range of wavelengths, called **long waves**, which is extended due to the optimization. The spatial discretization is suitable for waves resolved down to 5.4 points or more per wavelength for the DRP presented by Tam [1]. The whole method consisting of time integration and spatial discretization is unstable, if the waves reach the very short wave range, called **parasite waves** [2]. Between parasite and long waves **dispersive waves** are found, which are characterized by the high dependency of the group velocity to the wavenumber. The system of spatial and time discretization allows unphysically solutions which are the eigensolutions of the discretized (FDE), but not of the PDE. The short wave range is defined due to the fact, that waves are amplified in the time by the explicit one sided time discretization, if they reach this range.

The concept presented by Tam [2] to overcome this behavior is first to avoid this wave number range, and then to use a selective damping. This should disallow the waves to propagate below a certain resolution. Following the idea of Tam this concept can be seen as a dissipative term in the Euler equation for the conservation of momentum. The selectivity is obtained by optimizing the filtering coefficients to fit a Gaussian distribution in the dissipation over wavenumber diagram. However, the disadvantages of the concept are defined by the logical order of operations and the large dissipation, as we will see later.

10.2 Solution – Filtering

First we have to think about the order of operations. The selective damping is very strange to handle. Because the unfiltered field is used to compute the flux in the Euler Equations the new field is filtered only in parts and the filtering amplitude has to be adjusted to the typical frequency! Thus the filtering amplitude has to be adjusted for each case and the range of values differs from case to case. However, in principle the damping applies in every step and the new field is filtered. With the correct filtering amplitude the overall method remains stable. The dissipation of the optimized filter in the long wave range is too large! With a method optimized for low dissipation for short waves one could not accept a dissipation in the range of 3×10^{-4} applied in each time step even for a resolution of 32 ppw.

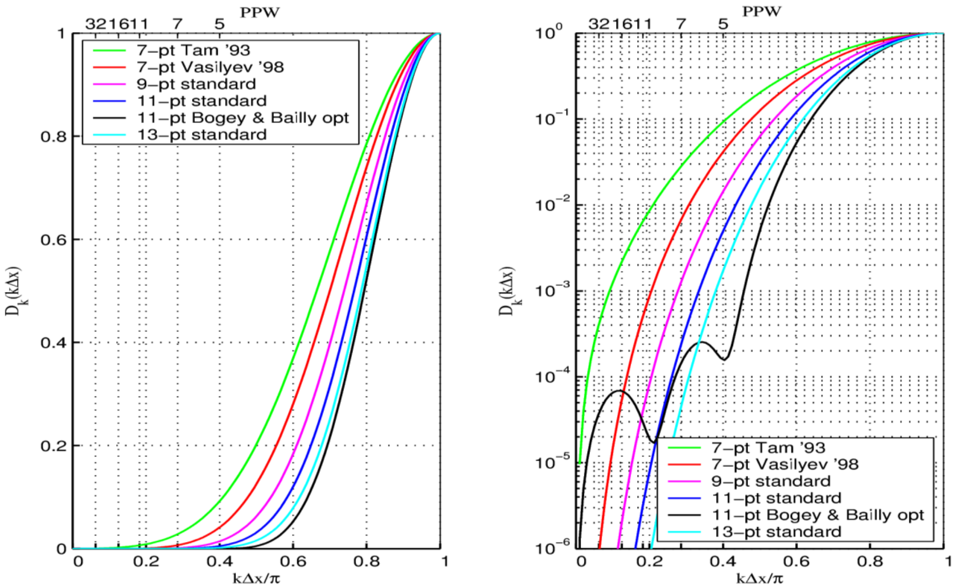


Figure 10.2: Filtering characteristics of different filter stencils

Now use the point of criticism to develop a new method. The filter should work on the field variables p' in order to find a new filtered field f'.See fig. 10.3 for the position for the filter.

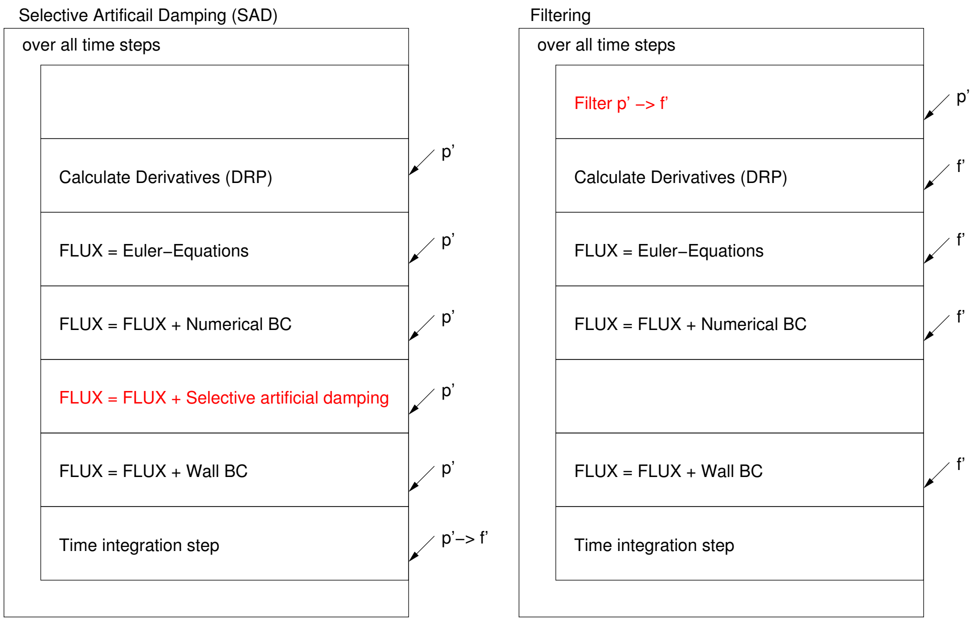


Figure 10.3: Selective artificial damping and filtering implementations in comparison (f' denotes the filtered, p' the original field)

Because the whole method we are developing up to this point is based on explicit finite difference approximations the filter should also be explicit.

10.3 Derivation of a Taylor filter

Let's say we find an approximation $\bar{\phi}$ for the value of a field variable ϕ in point n which is based on $-N$ to M neighboring points. $\bar{\phi}$ is supposed to be the filtered field and ϕ is the original field. w_j are the filter coefficients.

$$\bar{\phi}_n = \sum_{j=-N}^{M} w_j \, \phi_{n+j} \tag{10.1}$$

The error of such an filter in the wavenumber space is given by the answer to a harmonic excitation in space with the relative wavenumber $k\Delta x$:

$$\Psi = \sum_{j=-N}^{M} w_j \, \exp[i \, j \, k\Delta x] = \sum_{j=-N}^{M} w_j \, \cos[j \, k\Delta x] + i \sum_{j=-N}^{M} w_j \, \sin[j \, k\Delta x] \tag{10.2}$$

From (10.2) it can be observed, that a zero imaginary part would be achived by a symmetric set of filtering coefficients ($M = N$, $w_j = w_{-j}$) as the imaginary part is produced by a sine function, which is anti-metric around zero. A symmetric distribution of points involved into the filtering will allow zero imaginary parts and therefore no phase change:

$$w_j = w_{-j} \tag{10.3}$$

Therefore we concentrate on these central filter stencils with $M = N$ and consider (10.3) as the first condition to our filtering stencil.

The way to find a good approximation to a given field, which includes no short waves is a Taylor-series expansion. This technique is a standard for the filter design since digital filters are developed. The idea is, that the Taylor-series expansion will give an approximation by the neighboring points for the functional value in a given position, which is smooth to a certain order given by the number of points N. Therefore we develop (10.1) into a Taylor series around x_n:

$$\begin{aligned}
\bar{\phi}_n = {} & w_{-N} \left[\phi_n + \frac{\partial \phi_n}{\partial x} (-N \, \Delta x) + \frac{1}{2!} \frac{\partial^2 \phi_n}{\partial x^2} (-N \, \Delta x)^2 + \frac{1}{3!} \frac{\partial^3 \phi_n}{\partial x^3} (-N \, \Delta x)^3 \right] \\
& + w_{-N+1} \left[\phi_n + \frac{\partial \phi_n}{\partial x} ((-N+1) \, \Delta x) + \frac{1}{2!} \frac{\partial^2 \phi_n}{\partial x^2} ((-N+1) \, \Delta x)^2 + \frac{1}{3!} \frac{\partial^3 \phi_n}{\partial x^3} ((-N+1) \, \Delta x)^3 \right] \\
& \vdots \\
& + w_0 \, [\phi_n] \\
& \vdots \\
& + w_{N-1} \left[\phi_n + \frac{\partial \phi_n}{\partial x} ((N-1) \, \Delta x) + \frac{1}{2!} \frac{\partial^2 \phi_n}{\partial x^2} ((N-1) \, \Delta x)^2 + \frac{1}{3!} \frac{\partial^3 \phi_n}{\partial x^3} ((N-1) \, \Delta x)^3 \right] \\
& + w_N \left[\phi_n + \frac{\partial \phi_n}{\partial x} (N \, \Delta x) + \frac{1}{2!} \frac{\partial^2 \phi_n}{\partial x^2} (N \, \Delta x)^2 + \frac{1}{3!} \frac{\partial^3 \phi_n}{\partial x^3} (N \, \Delta x)^3 \right] \\
& + \mathcal{O}(\Delta x^4)
\end{aligned} \tag{10.4}$$

In a more general manner we could say that with a number of points in a distance $j \, \Delta x$ ($j = -N \ldots N$) from the point we are looking for a Taylor series denoted by the infinite sum over $k = 0 \ldots m$:

$$\bar{\phi}_n = \sum_{j=-N}^{N} w_j \left[\sum_{k=0}^{m} \frac{1}{k!} \frac{\partial^k \phi_n}{\partial x^k} (j \, \Delta x)^k \right] + \mathcal{O}(\Delta x^{m+1}) \tag{10.5}$$

After reordering the components with respect to the order of accuracy we can reach we obtain:

$$
\begin{aligned}
\bar{\phi}_n = & \left[w_{-N} + w_{-N+1} + \ldots + w_0 + \ldots + w_{N-1} + w_N\right] \phi_n \\
& + \left[w_{-N}\,(-N) + w_{-N+1}\,(-N+1) + \ldots + w_0\,0 + \ldots + w_{N-1}\,(N-1) + w_N\,(N)\right] \frac{\partial\,\phi_n}{\partial\,x} \\
& + \left[w_{-N}\,(-N)^2 + w_{-N+1}\,(-N+1)^2 + \ldots + w_0\,0 + \ldots + w_{N-1}\,(N-1)^2 + w_N\,(N)^2\right] \frac{\partial^{\,2}\phi_n}{\partial\,x^2} \\
& + \left[w_{-N}\,(-N)^3 + w_{-N+1}\,(-N+1)^3 + \ldots + w_0\,0 + \ldots + w_{N-1}\,(N-1)^3 + w_N\,(N)^3\right] \frac{\partial^{\,3}\phi_n}{\partial\,x^3} \\
& + \mathcal{O}(\Delta x^4)
\end{aligned}
$$

Reordered with respect to the finite sums and the summation indices j and k this equation reads:

$$
\bar{\phi}_n = \sum_{k=0}^{m} \frac{\Delta x^k}{k!} \frac{\partial^{\,k}\phi_n}{\partial\,x^k} \left[\sum_{j=-N}^{N} (j)^k\, w_j\right] + \mathcal{O}(\Delta x^{m+1}) \tag{10.6}
$$

If the inner sum is zero respectively, the summand of the Taylor series is zero. To get the approximation of the function value itself, the zero order must be multiplied by one. Together we get the following conditions:

$$
\sum_{j=-N}^{N} w_j = 1 \tag{10.7}
$$

$$
\sum_{j=-N}^{N} (j)\, w_j = 0 \tag{10.8}
$$

$$
\sum_{j=-N}^{N} (j)^2\, w_j = 0 \tag{10.9}
$$

$$
\vdots
$$

$$
\sum_{j=-N}^{N} (j)^{2(N-1)}\, w_j = 0 \tag{10.10}
$$

The symmetry condition (10.3) ensures a zero imaginary part and reduces the number of unknown filter coefficients by N. All odd exponentials multiplied on the filter coefficients are satisfied due to symmetry. Therefore only the even exponents give a new condition to the w_j. From (10.6) we can find only N of the $N+1$ conditions to the unknown filter coefficients. (10.7) to (10.10) plus the symmetry condition (10.3) do not fully fix the set of filter coefficients. However, the further conditions to reach a higher approximation order can not be fulfilled. It would conflict the conditions formulated before due to the additional condition given by the symmetry. The number of conditions will extend the number of unknown coefficients. With other words: we have to add points to achive a higher order. With a $2N+1$ point filtering stencil only a $2N^{th}$ order Taylor approximation to the value of a function itself can be achived.

The last condition to fix the set of coefficients is found by the filtering characteristics we would like to achive. The short wave component should be deleted from the given field fully, therefore we take the shortest resolvable wave and say that the answer of the filter to this input should be zero. The shortest possible wave resolved by only 2 ppw is a point to point grid oscillation (see figure fig. 10.4) which is to be deleted:

$$0 = \sum_{j=-N}^{N} (-1)^j \, w_j \tag{10.11}$$

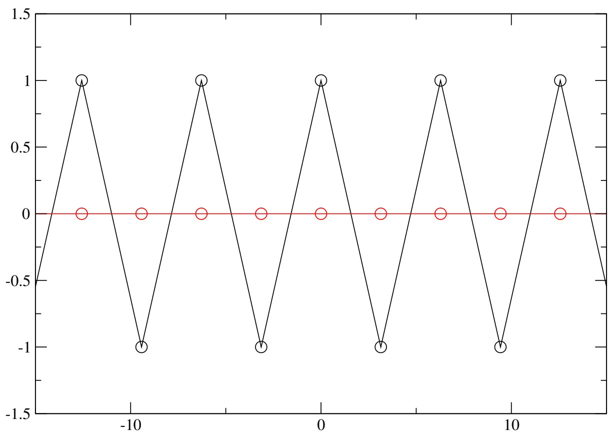


Figure 10.4: Spatial point to point oscillations (black circles), which should be deleted (red circles)

10.4 Conditions to the filter coefficients

Altogether, the conditions to an $2N^{th}$ order symmetric filter with $2N+1$ coefficients reads as follows:

$$\begin{aligned}
\sum_{j=-N}^{N} w_j &= 1 \\
\sum_{j=-N}^{N} (j)^2 \, w_j &= 0 \\
\sum_{j=-N}^{N} (j)^4 \, w_j &= 0 \\
&\vdots \\
\sum_{j=-N}^{N} (j)^{2(N-1)} \, w_j &= 0 \\
w_j &= w_{-j} \\
\sum_{j=-N}^{N} (-1)^j \, w_j &= 0
\end{aligned}$$

Hint:

In some publications the filter is defined as a change with respect to the original value:

$$\bar{\phi}_n = \phi_n + \sum_{j=-N}^{M} w_j \, \phi_{n+j} \tag{10.12}$$

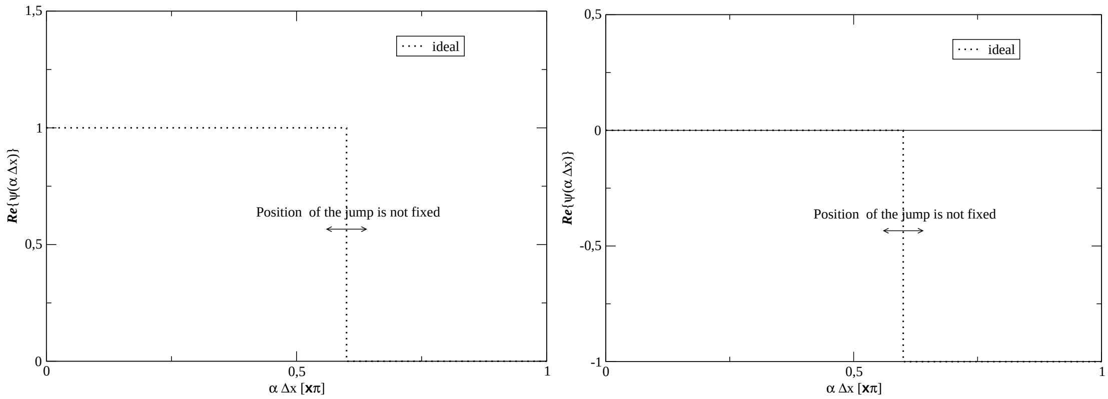


Figure 10.5: Idealized filter characteristics of both definitions

Due to the different definition we end up with a similar system, but the filter characteristics are reversed. The sum for the order zero must equal zero now, as the higher orders do. The answer to an point to point oscillation would be the negative value at the considered position, so that we get $\bar{\phi}_n = \phi_n - \phi_n = 0$ in this case:

$$\begin{aligned}
\sum_{j=-N}^{N} w_j &= 0 \\
\sum_{j=-N}^{N} (j)^2 \, w_j &= 0 \\
\sum_{j=-N}^{N} (j)^4 \, w_j &= 0 \\
&\vdots \\
\sum_{j=-N}^{N} (j)^{2(N-1)} \, w_j &= 0 \\
w_j &= w_{-j} \\
\sum_{j=-N}^{N} (-1)^j \, w_j &= -1
\end{aligned}$$

The advantage of this definition is, that it allows a little filtering and it is directly applicable as SAD in order to compare both methods directly. However, this kind of definition is contrary to any publication on digital filters.

10.5 Filter coefficients

Table 10.1: Filter coefficients for a filter defined to be $\bar{\phi}_n = \phi_n + \sum_{j=-N}^{M} w_j \, \phi_{n+j}$

$w_{-6} = w_6$	$w_{-5} = w_5$	$w_{-4} = w_4$	$w_{-3} = w_3$	$w_{-2} = w_2$	$w_{-1} = w_1$	w_0	$\mathcal{O}(\Delta x^k)$
-	-	-	-	-	$\frac{1}{4}$	$-\frac{1}{2}$	$\mathcal{O}(\Delta x^2)$
-	-	-	-	$-\frac{1}{16}$	$\frac{1}{4}$	$-\frac{3}{8}$	$\mathcal{O}(\Delta x^4)$
-	-	-	$\frac{1}{64}$	$-\frac{3}{32}$	$\frac{15}{64}$	$-\frac{5}{16}$	$\mathcal{O}(\Delta x^6)$
-	-	$-\frac{1}{256}$	$\frac{1}{32}$	$-\frac{7}{64}$	$\frac{7}{32}$	$-\frac{35}{128}$	$\mathcal{O}(\Delta x^8)$
-	$\frac{1}{1024}$	$-\frac{5}{512}$	$\frac{45}{1024}$	$-\frac{15}{128}$	$\frac{105}{512}$	$-\frac{63}{256}$	$\mathcal{O}(\Delta x^{10})$
$-\frac{1}{4096}$	$\frac{3}{1024}$	$-\frac{33}{2048}$	$\frac{55}{1024}$	$-\frac{495}{4096}$	$\frac{99}{512}$	$-\frac{231}{1024}$	$\mathcal{O}(\Delta x^{12})$

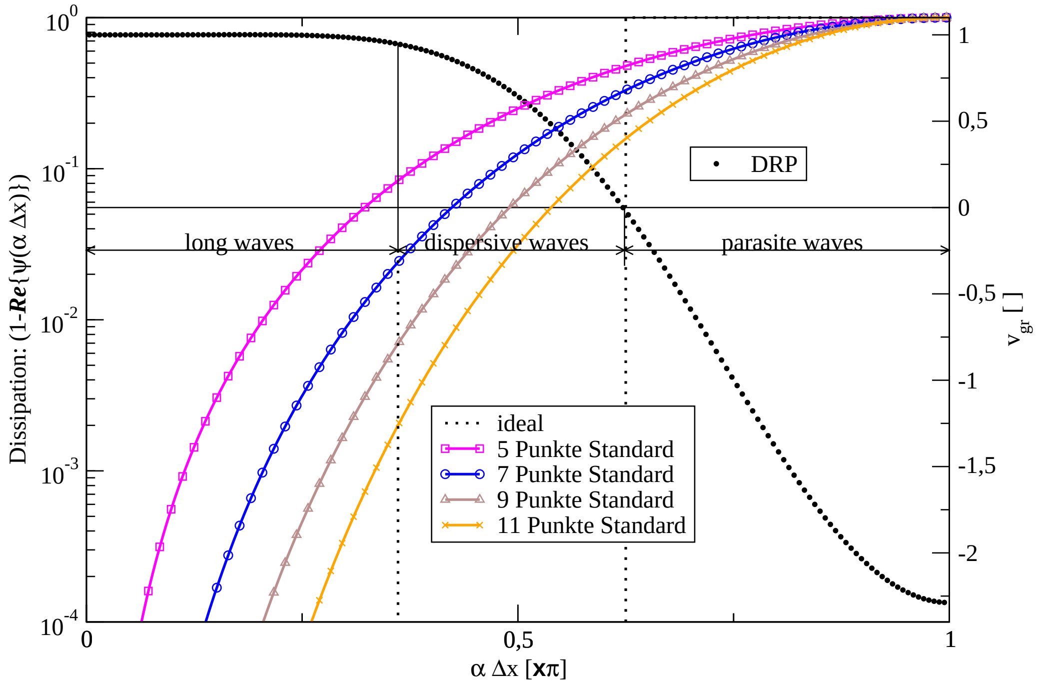


Figure 10.6: Different standard filters and the DRP depending on $\alpha\Delta x$

The more points are used for the filter, the wider is the bandwith of the passed wavelengths, i.e. a small filter could filter too much waves.

10.6 Optimized filters

There are several publications on optimized filters, but they all suffer from the problem, that the approximation characteristics in the error over wavenumber graph is made of only N cosine functions. A optimization in order to move or steepen the cut off from pass band to blocked wavelengths will therefore also lead to a larger error in the pass band. The Taylor series is in an optimum if the dissipation over the pass band would be the criterion.

To achive better filter characteristics one would have to use more points in the filter stencil. A comparison of the filter and the DRP characteristics helps to identify the 11 point filter as the best approximation to the ideal filter curve.

10.7 Comparison SAD vs. Filter

Compared to the selective artificial damping, the filter is less dissipative in the pass band. The use of the modified filter formula allows to give a filter amplitude to control the dissipation. The direct change of the field variables allows arbitrary adjustment of the filter amplitude. The filter coefficients could even be used in a selective damping way.

The placement outside allows to give filter cycles. Filtering only every T^{th} timestep allows still to delete the short and dispersive waves, with at the same time even lower dissipation. The computational performance of such an cycled filter even with higher point number is better than that of the SAD.

References

[1] C. K. W. Tam and C. Webb. Dispersion-Relation-Preserving Finite Difference Schemes for Computational Aeroacoustics. *Journal of Computational Physics*, 107(2):262–281, August 1993.

[2] C. K. W. Tam, C. Webb, and T. Z. Dong. A Study of Short Wave Components in Computational Aeroacoustics. *Journal of Computational Acoustics*, 1:1–30, März 1993.

11 Wall Boundary Conditions for High-Order Finite Difference Schemes

For a higher order finite difference scheme the order of the difference equations is higher than that of the Euler equations. Therefore zero normal velocity boundary condition (wall boundary condition) is not sufficient to define a unique solution. Additional conditions must be imposed. Unfortunately these additional conditions would inevitably lead to the generations of the spurious numerical waves. A special care (boundary condition) is needed.

11.1 Definitions

Boundary points:	the first three rows of points adjacent to the wall
Interior points:	points lying three rows or more away from the wall
Ghost points:	points outside the wall (the computational domain)

11.2 Why do we need the ghost points?

Fact
In the discretized system each flow variable at interior and boundary points is governed by an algebraic equation (FDE). Thus the number of unknowns is exactly equal to the number of equations.

Problem
There will be too many equations and not enough unknowns if we would like to enforce the boundary conditions at the wall.

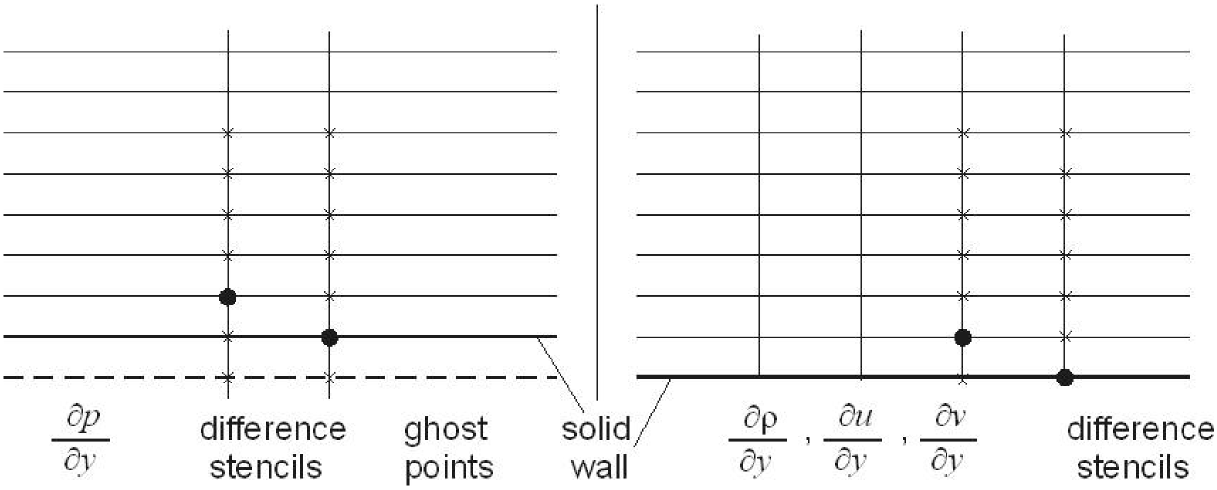
Solution
By introducing ghost points, the additional conditions imposed on the flow variables by the wall boundary conditions can be satisfied. The number of ghost points must be equal to the number of boundary conditions.

Let us consider an inviscid fluid flow condition, the wall boundary condition is $v = 0$ and $\frac{\partial \vec{v}'_n}{\partial t}$ at y=0 where (u, v) are the velocity components in the x and y direction respectively. We have one boundary condition; one ghost value is therefore needed for each boundary point at the wall. The actual physical process suggests that a ghost value in pressure to be considered:

Linearized Momentum Equation:

$$\begin{aligned}
\frac{\partial \vec{v}'}{\partial t} &= -\vec{f}(\vec{v}, \text{grad}\vec{v}, \ldots) - \frac{1}{\rho_0}\text{grad}p' && \Big|\cdot\vec{n} \\
\frac{\partial \vec{v}'_n}{\partial t} &= -\vec{n}\vec{f}(\vec{v}, \text{grad}\vec{v}, \ldots) - \frac{1}{\rho_0}\frac{\partial p'}{\partial \vec{n}} && \Big|\frac{\partial \vec{v}'_n}{\partial t} = 0 \text{ with wall boundary condition} \\
\Rightarrow \frac{\partial p'}{\partial \vec{n}} &= -\rho_0 \cdot \vec{n} \cdot \vec{f}(\vec{v}, \ldots)
\end{aligned}$$

7-point stencils used for computation



stencils used for $\frac{\partial p}{\partial y}$ stencils used for $\frac{\partial \rho}{\partial y}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial y}$

11.3 Implementation

The 7-point-stencil DRP scheme:

$$\vec{K}^{(n)}_{l,m} = -\frac{1}{\Delta x}\sum_{j=-3}^{3} a_j\, \vec{E}^{(n)}_{l+j,m} - \frac{1}{\Delta y}\sum_{j=-3}^{3} a_j\, \vec{F}^{(n)}_{l,m+j} + \vec{H}^{(n)}_{l,m}$$

$$\vec{Q}^{(n+1)}_{l,m} = \vec{Q}^{(n)}_{l,m} + \Delta t \sum_{j=0}^{3} b_j\, \vec{K}^{(n-j)}_{l,m}$$

The above scheme needs to be modified for the third element v of the discretized Euler equations at the boundary ($y = 0$). Assuming $m = 0$ at $y = 0$, we have

$$\boxed{v^{(n+1)}_{l,0} = v^{(n)}_{l,0} + \Delta t \sum_{j=0}^{3} b_j K^{(n-j)}_{l,0}} \tag{11.1}$$

where $K^{n-j}_{l,0} = \left(\frac{dv}{dt}\right)^{n-j}_{l,0}$.

From the third equation of the Euler equations, $\frac{\partial v}{\partial t} + u_0 \frac{\partial v}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}$ we have

$$\boxed{K^{n-j}_{l,0} = -\frac{u_0}{\Delta x}\sum_{i=-3}^{3} a_i v^{(n-j)}_{l+i,0} - \frac{1}{\rho_0 \Delta y}\sum_{i=-1}^{5} a^{15}_i p^{(n-j)}_{l,i}} \tag{11.2}$$

where the backward stencil coefficient a^{15}_i means, that there is one point forwards and five points backwards. The ghost value $p^{(n)}_{l,-1}$ is found by setting $v^{(n+1)}_{l,0} = 0$, $v^0_{l,n} = 0$ in equation (11.1). Then we have $K^{(n-j)}_{l,0} = 0$. From equation (11.2) with $v^{(n-j)}_{l+i,0}$ we can determine the ghost value

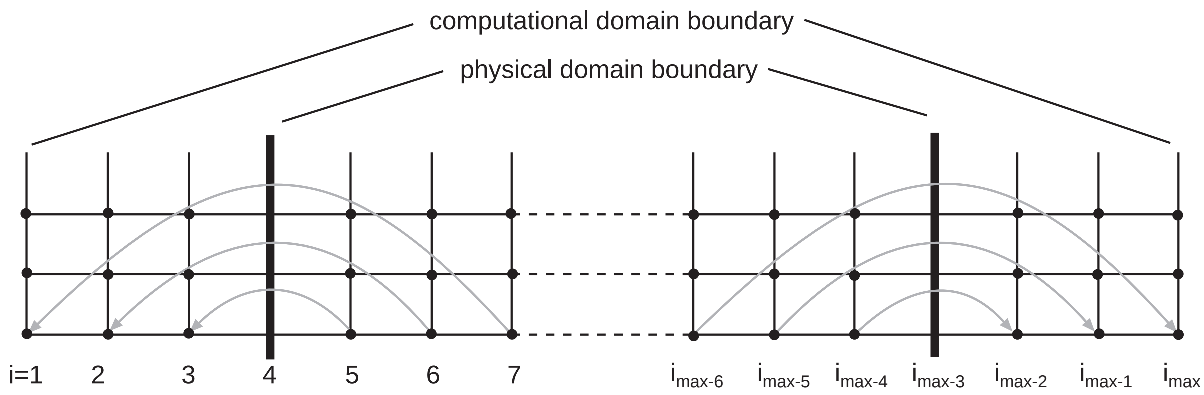
$$p^{(n)}_{l,-1} = -\frac{1}{a^{15}_{-1}} \sum_{i=0}^{5} a^{15}_i p^{(n)}_{l,i}$$

which is equivalent to set the ghost value such that $\frac{\partial p}{\partial y} = 0$ at the wall.

For viscous fluid flow, no-slip boundary condition is $u = v = 0$. The ghost value $p_{l,-1}^{(n)}$ ensures the condition $v_{l,0}^{(n+1)} = 0$ is satisfied at the wall. But the wall also exerts a shear stress τ_{xy} on the fluid to reduce the velocity component u to zero ($u = 0$). Another ghost value $(\tau_{xy})_{l,-1}^{(n)}$ is needed to ensure that $u_{l,0}^{(n+1)} = 0$. Therefore the same backward stencils as for the pressure are used for $\partial\tau_{xy}/\partial y$ in the x-momentum equation.

11.4 Symmetric Boundary Conditions

For symmetric boundary conditions, the values for pressure and velocity are mirrored on the plane of symmetry. Consider a central 7-point-stencil:



Boundary on the left:

$$\begin{aligned} u_i &= -u_{8-i} \\ p_i &= p_{8-i} \end{aligned}$$

Boundary on the right:

$$\begin{aligned} u_i &= -u_{i_{max}-6+(i_{max}-i)} \\ p_i &= p_{i_{max}-6+(i_{max}-i)} \end{aligned}$$

12 Non-reflective Boundary Conditions

Since the DRP scheme and the PDE have the same dispersion relations in the limits of $\bar{\alpha} \simeq \alpha$, $\bar{\beta} \simeq \beta$ and $\bar{\omega} = \omega$, the radiation and the Outflow boundary conditions can be constructed from the asymptotic solutions obtained by the Fourier-Laplace analysis of the Euler equations.

12.1 Radiation Boundary Conditions (Two-Dimension)

The asymptotic behavior of the acoustic waves is

$$\begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix}_a = \frac{F\left(\frac{r}{V(\theta)} - t, \theta\right)}{r^{\frac{1}{2}}} \begin{bmatrix} \frac{1}{a_a^2} \\ \frac{\hat{u}(\theta)}{\rho_0 c_0} \\ \frac{\hat{V}(\theta)}{\rho_0 c_0} \\ 1 \end{bmatrix} + \mathcal{O}\left(r^{-\frac{3}{2}}\right) \tag{12.1}$$

acoustic waves (polar coordinates r, θ)

where

$$\begin{aligned} V(\theta) &= c_0 \left[M \cos\theta + \sqrt{1 - M^2 \sin^2\theta}\right] \\ \hat{u}(\theta) &= \cos\theta - M\sqrt{1 - M^2 \sin^2\theta} \\ \hat{V}(\theta) &= \sin\theta \left[\sqrt{1 - M^2 \sin^2\theta} + M\cos\theta\right] \end{aligned}$$

By taking the partial derivatives of equation (12.1) (first component) with respect to the time t and the spatial coordinate r

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \frac{-F'\left(\frac{r}{V(\theta)} - t, \theta\right)\left(\frac{1}{c_0^2}\right)}{\sqrt{r}} + \mathcal{O}\left(r^{-\frac{3}{2}}\right) \\ \frac{\partial \rho}{\partial r} &= \frac{\frac{\sqrt{\gamma}}{V(\theta)} F'\left(\frac{r}{V(\theta)} - t, \theta\right) - \frac{1}{2\sqrt{r}} F\left(\frac{r}{V(\theta)} - t, \theta\right)}{r} \left[\frac{1}{c_0^2}\right] + \mathcal{O}\left(r^{-\frac{5}{2}}\right) \end{aligned}$$

We have

$$\frac{1}{V(\theta)} \cdot \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial r} + \frac{\rho}{2r} = 0 + \mathcal{O}\left(r^{-\frac{5}{2}}\right)$$

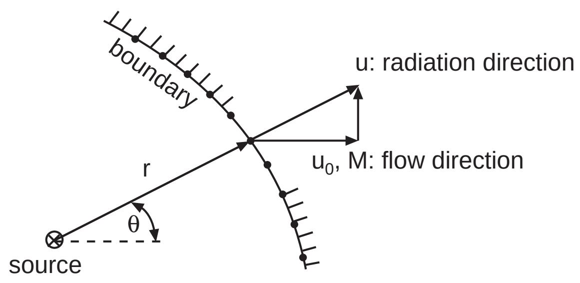
Similarly, we can derive for the far field

$$\frac{1}{V(\theta)} \cdot \frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{1}{2r} \begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix}_a = 0 + \mathcal{O}\left(r^{-\frac{5}{2}}\right) \tag{12.2}$$

θ is the angular coordinate of the boundary point and in Cartesian coordinates

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}$$

To solve the equations at the boundaries, the location of the source Q has to be determined and then the radiuses and angles have to be evaluated for every boundary point.



Equation (12.2) provides a set of radiation boundary conditions without flow exiting the domain (no incoming acoustic waves). With incoming acoustic waves and given ρ_{in}, u_{in}, v_{in} and p_{in} we have

$$\left(\frac{1}{V(\theta)} \cdot \frac{\partial}{\partial t} + \frac{\partial}{\partial r} + \frac{1}{2r}\right) \begin{bmatrix} \rho - \rho_{in} \\ u - u_{in} \\ v - v_{in} \\ p - p_{in} \end{bmatrix}_a = 0 + \mathcal{O}\left(r^{-\frac{5}{2}}\right)$$

Note on the radiation boundary condition

1. Accurate and efficient in even moderate far field
2. Not suitable near acoustic sources or when the mean flow is very non-uniform.

12.2 Non reflective Layer boundary conditions

12.2.1 Termination of the computational domain by a simple sponge layer

We think about the 2D LEE written in terms of:

$$\frac{\partial \underline{q}}{\partial t} = \underbrace{-\underline{A} \cdot \frac{\partial \underline{q}}{\partial x} - \underline{B} \cdot \frac{\partial \underline{q}}{\partial y} - \underline{D} \cdot \underline{q}}_{F_{phys.}} \tag{12.3}$$

With

$$\underline{q} := \left(\varrho', \quad u', \quad v', \quad w', \quad p'\right)^T \tag{12.4}$$

and the derivatives of $\underline{q}$ in x are multiplied by:

$$\underline{A} := \begin{pmatrix} \bar{U} & \bar{\varrho} & 0 & 0 \\ 0 & \bar{U} & 0 & \frac{1}{\bar{\varrho}} \\ 0 & 0 & \bar{U} & 0 \\ 0 & \gamma \bar{P} & 0 & \bar{U} \end{pmatrix} \tag{12.5}$$

The derivatives in y are multiplied by:

$$\underline{B} := \begin{pmatrix} \bar{V} & 0 & \bar{\varrho} & 0 & \\ 0 & \bar{V} & 0 & 0 & \\ 0 & 0 & \bar{V} & \frac{1}{\bar{\varrho}} & \\ 0 & 0 & \gamma \bar{P} & 0 & \bar{V} \end{pmatrix} \tag{12.6}$$

The derivatives in the mean flow are for example given by:

$$\underline{D} := \begin{pmatrix} \frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial r} & \frac{\partial \bar{\varrho}}{\partial x} & \frac{\partial \bar{\varrho}}{\partial r} & 0 \\ \frac{1}{\bar{\varrho}}(\bar{U}\frac{\partial \bar{U}}{\partial x} + \bar{V}\frac{\partial \bar{U}}{\partial r}) & \frac{\partial \bar{U}}{\partial x} & \frac{\partial \bar{U}}{\partial r} & 0 \\ \frac{1}{\bar{\varrho}}(\bar{U}\frac{\partial \bar{V}}{\partial x} + \bar{V}\frac{\partial \bar{V}}{\partial r}) & \frac{\partial \bar{V}}{\partial x} & \frac{\partial \bar{V}}{\partial r} & 0 \\ 0 & \frac{\partial \bar{P}}{\partial x} & \frac{\partial \bar{P}}{\partial r} & -\frac{1}{\bar{P}}\left[\bar{U}\frac{\partial \bar{P}}{\partial x} + \bar{V}\frac{\partial \bar{P}}{\partial r}\right] \end{pmatrix} \tag{12.7}$$

Then a simple sponge layer with proportional damping is defined by:

$$\frac{\partial \underline{q}}{\partial t} = -\underline{F}_{phys.}(\underline{q}) - R_d(x,r)\,(\underline{q} - \underline{q}_0) \tag{12.8}$$

The proportional damping will dissipate any wave traveling towards the boundary region of the domain. The formula is strongly oriented on the mechanical formulas of proportional damping. This kind of so called Newton cooling/friction type damping in the sponge layer was first introduced by Israeli [5]. Producing a simple 1D wave equation from equation (12.8) we start with:

$$\frac{\partial p'}{\partial t} + c^2\,\bar{\varrho}\,\frac{\partial u'}{\partial x} + R_d\,p' = 0 \tag{12.9}$$

$$\frac{\partial u'}{\partial t} + \frac{1}{\bar{\varrho}}\frac{\partial p'}{\partial x} + R_d\,u' = 0 \tag{12.10}$$

Now like in the derivation of the wave equation we derive the equations in time and space respectively:

$$\frac{\partial^2 p'}{\partial t^2} + c^2\,\bar{\varrho}\,\frac{\partial}{\partial t}\frac{\partial u'}{\partial x} + \frac{\partial R_d\,p'}{\partial t} = 0 \tag{12.11}$$

$$\bar{\varrho}\,\frac{\partial}{\partial x}\frac{\partial u'}{\partial t} + \frac{\partial^2 p'}{\partial x^2} + \bar{\varrho}\,\frac{\partial R_d\,u'}{\partial x} = 0 \tag{12.12}$$

The wave equation is:

$$\frac{\partial^2 p'}{\partial t^2} + \frac{\partial R_d\,p'}{\partial t} = c^2\,\bar{\varrho}\,\frac{\partial R_d\,u'}{\partial x} + c^2\,\frac{\partial^2 p'}{\partial x^2} \tag{12.13}$$

Pressure and velocity are not fully decoupled, but one can observe, that a damping first order derivative is added in the time. If we replace $c^2\,\bar{\varrho}\,\frac{\partial R_d\,u'}{\partial x}$ we can also see, that a damping first order derivative in space is added to the right hand side.

$$\frac{\partial^2 u'}{\partial t^2} + \frac{\partial R_d\,u'}{\partial x} = \frac{1}{\bar{\varrho}\,c^2}\frac{\partial R_d\,p'}{\partial x} + c^2\,\frac{\partial^2 u'}{\partial x^2} \tag{12.14}$$

Even though the partial differential equation is not solved, we can conclude, that this kind of damping will reduce the amplitude of waves traveling in any direction.

The main disadvantage is that the sponge layer will reflect waves, as the wave numbers inside the interior domain and inside the sponge layer do not match. This may be overcome by introducing a distribution of the damping coefficient, that shifts the main change of wave number to a region inside the sponge layer (compare fig. 12.1):

$$R_d(x,r) = \begin{cases} \exp\left\{-\frac{1}{2}\,n_P\,\frac{d_{BC}^2}{\Delta x_{NC/F}^2}\right\} & , \quad d_{BC} < \Delta x_{NC/F} \\ 0 & , \quad \text{else} \end{cases} \tag{12.15}$$

To further improve the performance the sponge layer could be stretched towards the boundary, but stretching ratio should not exceed 1.05. Above the short wave generation will annihilate the advantages of the stretching. Selective Damping or filtering is additionally highly recommended with grid stretching. The boundary condition is extremely simple, but requires a extension of the domain. It works close to sources, as well as with numerical artefacts. Due to the short wavelength the parasite waves are damped very efficiently. The boundary condition is independent of the used physics and works with LEE as well as APE or PENNE type equations. However, as the damping is not selective a mismatch of wave numbers may lead to reflections. That is the reason to extend this simple "(sponge) layer" boundary condition by a more thought over one. In case of sheared flow or instability waves, this kind of boundary condition is unstable even.

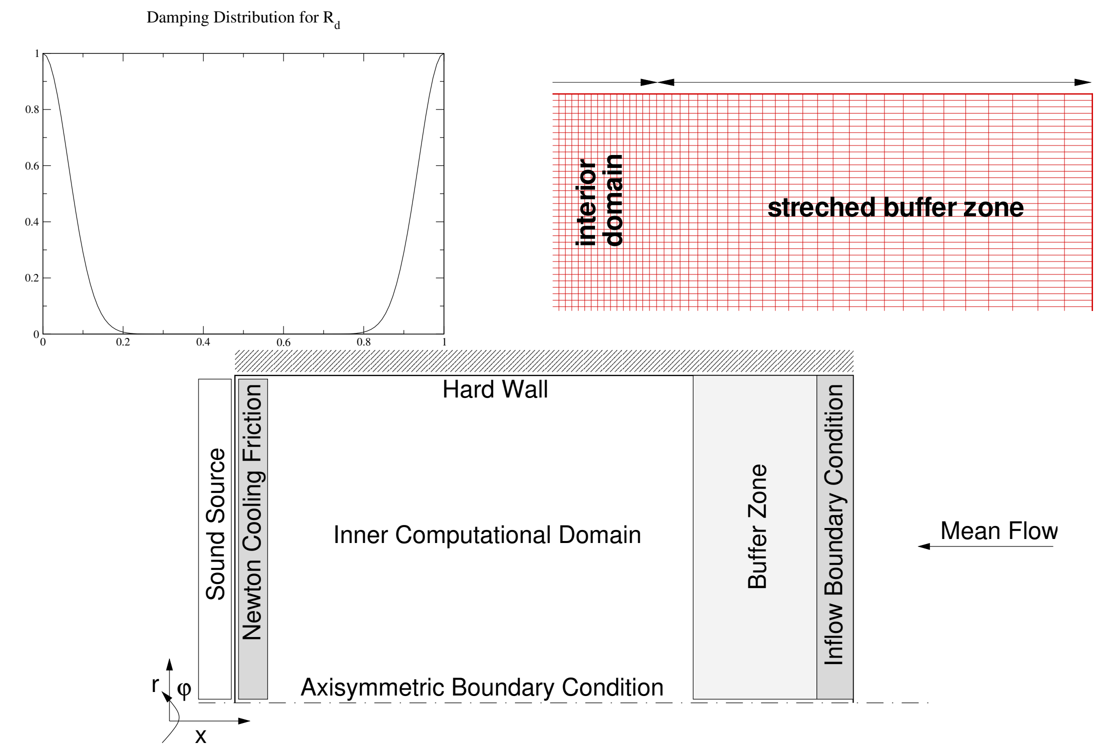


Figure 12.1: Damping distribution and stretched axisymmetric grid

12.2.2 Split formulation of the PML

In the frequency domain a useful technique to produce non–reflective boundary conditions is to introduce a imaginary part to the frequency. The perfect matching is reached by keeping the dispersion relation constant, and only adding a imaginary part to the frequency and the related wave number pointing towards the outer boundary of the PML. As only the wave components impacting normal to the end of the computational domain should be damped, the developers of the PML introduced a split of variables. Even the earliest PML for the electrodynamics from Berenger [1] used this kind of derivation. For an arbitrary vector $\underline{q}$ this reads in the frequency domain:

$$\begin{aligned} -i\,\omega\underline{\hat{q}}_x + \sigma_x\,\underline{\hat{q}}_x &= -\underline{A}\cdot\frac{\partial\,\underline{\hat{q}}}{\partial\,x} \\ -i\,\omega\underline{\hat{q}}_y + \sigma_y\,\underline{\hat{q}}_y &= -\underline{B}\cdot\frac{\partial\,\underline{\hat{q}}}{\partial\,y} \end{aligned} \tag{12.16}$$

Please note that electrodynamics do not account for any convective effects and we use a $e^-i\omega\,t$ convention. This is the reason that the simple transcription of the so called split PML to the LEE by Hu [3] in the time domain is unstable if a mean flow is present [6].

$$\begin{aligned} \frac{\partial\,\underline{\hat{q}}_x}{\partial\,t} + \sigma_x\,\underline{\hat{q}}_x &= -\underline{A}\cdot\frac{\partial\,\underline{\hat{q}}}{\partial\,x} \\ \frac{\partial\,\underline{\hat{q}}_y}{\partial\,t} + \sigma_y\,\underline{\hat{q}}_y &= -\underline{B}\cdot\frac{\partial\,\underline{\hat{q}}}{\partial\,y} \end{aligned} \tag{12.17}$$

12.2.3 Unsplit formulation of the PML

There are two things solved by Hu [4]:

(a) The original PML is unstable under a mean flow

(b) The split of the variables has to be maintained in the whole computational domain, which doubles the number of variables in 2D and makes the equations not easy to implement.

(c) The mean flow derivatives are not yet accounted for.

2. Unsplitting the PML Let us first address the second point. The split of the variables can be undone following Hu [4]:

We multiply the split equations by $1 + \frac{i\sigma_x}{\omega}$ and $1 + \frac{i\sigma_y}{\omega}$ respectively:

$$\begin{aligned} -i\,\omega\big(1+\frac{i\,\sigma_x}{\omega}\big)\,\big(1+\frac{i\,\sigma_y}{\omega}\big)\,\underline{\hat{q}}_x &= -\big(1+\frac{i\,\sigma_y}{\omega}\big)\,\underline{A}\cdot\frac{\partial\,\underline{\hat{q}}}{\partial\,x} \\ -i\,\omega\big(1+\frac{i\,\sigma_x}{\omega}\big)\,\big(1+\frac{i\,\sigma_y}{\omega}\big)\,\underline{\hat{q}}_y &= -\big(1+\frac{i\,\sigma_x}{\omega}\big)\,\underline{B}\cdot\frac{\partial\,\underline{\hat{q}}}{\partial\,y} \end{aligned} \tag{12.18}$$

Then the equations are added up, and as the factor in front of the left hand side is equal, we can unsplit the left hand side as $\underline{\hat{q}} = \underline{\hat{q}}_x + \underline{\hat{q}}_y$:

$$\begin{aligned} -i\,\omega\big(1+\frac{i\,\sigma_x}{\omega}\big)\,\big(1+\frac{i\,\sigma_y}{\omega}\big)\,\underline{\hat{q}} &= \\ -\big(1+\frac{i\,\sigma_y}{\omega}\big)\,\underline{A}&\cdot\frac{\partial\,\underline{\hat{q}}}{\partial\,x} \\ -\big(1+\frac{i\,\sigma_x}{\omega}\big)\,\underline{B}&\cdot\frac{\partial\,\underline{\hat{q}}}{\partial\,y} \end{aligned} \tag{12.19}$$

Multiplying and sorting all the terms into the exponents of $-i\omega$ delivers the PML equation for one frequency. The back transformation into the time domain is done replacing the $-i\,\omega$ terms by time derivatives and the $\frac{i}{\omega}$ terms by time integrals. To allow this we introduce a new variable q_1, which is only defined and computed in the PML region:

$$\frac{\partial\,q_1}{\partial\,t} = \underline{\hat{q}} \tag{12.20}$$

One can see that the number of variables has not really reduced in the PML, but the implementation is simplified.

1. Stability to the PML The reason for the instability of the PML is the mean flow. It allows waves, that have positive group velocity, but negative phase velocity. The PML printed above is not well posed in this case. The whole idea works only, if there is phase and group velocity with equal signs. This is always true for the vorticity and entropy modes, but the acoustic waves produce errors. To get rid of the problem, a transformation to the moving frame of reference is introduced. This allows to use the original PML and it will be well posed by default. Then the backward transformation will give the stable PML formulation for the fixed frame.

3. Accounting for the mean flow This part is the simplest, as the mean flow can be seen as a source term. This source term changes wave numbers and produces reflections therefore. The PML is artificial. The sources in this region should not be accounted specially for the PML.

$$\underline{\hat{q}} = \underline{\hat{q}}_x + \underline{\hat{q}}_y + \underline{\hat{q}}_M \tag{12.21}$$

So simply do nothing additional, but keep this part in the LEE. The PML assumes a constant mean flow in the PML, which may lead to an ill posedness, if highly not satisfied. However, even shear layers and instability waves are absorbed by the unsplit PML.

Final PML formulation in 2D Finally we end with the PML of Hu in a very simple formulation:

$$\begin{aligned}\frac{\partial \underline{\hat{q}}}{\partial t} = & \underbrace{-\underline{A} \cdot \frac{\partial \underline{\hat{q}}}{\partial x} - \underline{B} \cdot \frac{\partial \underline{\hat{q}}}{\partial r} - \frac{1}{r}\underline{C} \cdot \underline{\hat{q}} - \underline{D} \cdot \underline{\hat{q}}}_{\text{Euler equations}} \\ & \underbrace{-(\sigma_x + \sigma_r + \sigma_\varphi)\, \underline{\hat{q}}}_{\text{sponge layer}} - (\sigma_x\, \sigma_y)\, \underline{q}_1 - \sigma_y\, \underline{A} \cdot \frac{\partial \underline{q}_1}{\partial x} - \sigma_x\, \underline{B} \cdot \frac{\partial \underline{q}_1}{\partial y} \\ & \underbrace{-\frac{M_x}{1 - M_x^2}\, \underline{A} \cdot \left[\sigma_x\, \underline{\hat{q}} + \sigma_x\, \sigma_y\, \underline{q}_1\right]}_{\text{fixed frame correction}}\end{aligned} \tag{12.22}$$

There are several different groups developing PML equations for the LEE e.g. Hesthaven [2]. It seems that there are perfectly matched several solutions, that depend on the definition of the auxiliary variable $\underline{q}_1$. As the PML is in fact a sponge layer with correction for the preservation of the dispersion relation, the damping coefficient could be distributed in the same way. The PML damping coefficient can however be adjusted more freely. The allowed range is large, and even though it should be smooth to minimize reflections, the PML can be very short compared to the sponge layer.

12.3 Outflow Boundary Conditions (Two-Dimension)

At the outflow boundary, it consists a combination of entropy, vorticity and acoustic waves. Therefore, if the flow points in positive x-direction outside the domain

$$\begin{bmatrix} \rho \\ u \\ v \\ p \end{bmatrix} = + \begin{bmatrix} f\,(x - u_0 t, y) + \rho_a \\ \frac{\partial \psi}{\partial y}\,(x - u_0 t, y) + u_a \\ -\frac{\partial \psi}{\partial y}\,(x - u_0 t, y) + v_a \\ p_a \end{bmatrix} + ... \tag{12.23}$$

As we can see that the outflow boundary condition for p is the same as that of the radiation boundary condition.

$$\frac{1}{V\,(\theta)}\frac{\partial p}{\partial t} + \frac{\partial p}{\partial r} + \frac{p}{2r} = 0$$

and in Cartesian coordinates,

$$\frac{1}{V\,(\theta)}\frac{\partial p}{\partial t} + \cos\theta\frac{\partial p}{\partial x} + \sin\theta\frac{\partial p}{\partial y} + \frac{p}{2r} = 0$$

The first component of (12.23) contains an additional term from the entropy. By taking the partial derivatives of equation (12.23) with respect to the time t and the spatial coordinate x

$$\begin{aligned}\frac{\partial \rho}{\partial t} &= -u_0 f'(x - u_0 t, y) + \frac{\partial \rho_a}{\partial t} \\ \frac{\partial \rho}{\partial x} &= f'(x - u_0 t, y) + \frac{\partial \rho_a}{\partial x}\end{aligned}$$

we have

$$\frac{\partial \rho}{\partial t} + u_0 \frac{\partial \rho}{\partial x} = \frac{\partial \rho_a}{\partial t} + u_0 \frac{\partial \rho_a}{\partial x}$$

Due to the fact that $p_a = p = c_0^2 \rho_a$ (the acoustic equation of state), $\rho_a$can be eliminated. The outflow boundary condition for ρ is therefore

$$\frac{\partial \rho}{\partial t} + u_0 \frac{\partial \rho}{\partial x} = \frac{1}{c_0^2}\left(\frac{\partial p}{\partial t} + u_0 \frac{\partial p}{\partial x}\right)$$

The second and third component of (12.23) contains additional terms from the vorticity. Following the same procedure, we have

$$\begin{aligned}\frac{\partial u}{\partial t} + u_0 \frac{\partial u}{\partial x} &= \frac{\partial u_a}{\partial t} + u_0 \frac{\partial u_a}{\partial x} \\ \frac{\partial v}{\partial t} + u_0 \frac{\partial v}{\partial x} &= \frac{\partial v_a}{\partial t} + u_0 \frac{\partial v_a}{\partial x}\end{aligned}$$

Since the acoustic components satisfy the linearized Euler equations

$$\begin{aligned}\frac{\partial u_a}{\partial t} + u_0 \frac{\partial u_a}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p_a}{\partial x} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\ \frac{\partial v_a}{\partial t} + u_0 \frac{\partial v_a}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p_a}{\partial y} = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}\end{aligned}$$

The outflow boundary conditions for the velocity components u and v are after eliminating u_a and v_a

$$\begin{aligned}\frac{\partial u}{\partial t} + u_0 \frac{\partial u}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\ \frac{\partial v}{\partial t} + u_0 \frac{\partial v}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y}\end{aligned}$$

The outflow boundary conditions for all the variables are

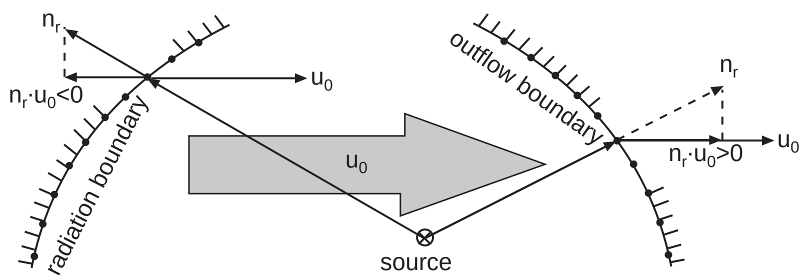
$$\begin{aligned}\frac{\partial \rho}{\partial t} + u_0 \frac{\partial \rho}{\partial x} &= \frac{1}{c_0^2}\left(\frac{\partial p}{\partial t} + u_0 \frac{\partial p}{\partial x}\right) \\ \frac{\partial u}{\partial t} + u_0 \frac{\partial u}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \\ \frac{\partial v}{\partial t} + u_0 \frac{\partial v}{\partial x} &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \\ \frac{1}{V(\theta)}\frac{\partial p}{\partial t} + \cos\theta \frac{\partial p}{\partial x} + \sin\theta \frac{\partial p}{\partial y} + \frac{p}{2r} &= 0\end{aligned} \qquad (12.24)$$

where $V(\theta)$ is the same as in equation (12.1).

12.4 Implementation of Radiation and Outflow Boundary Conditions

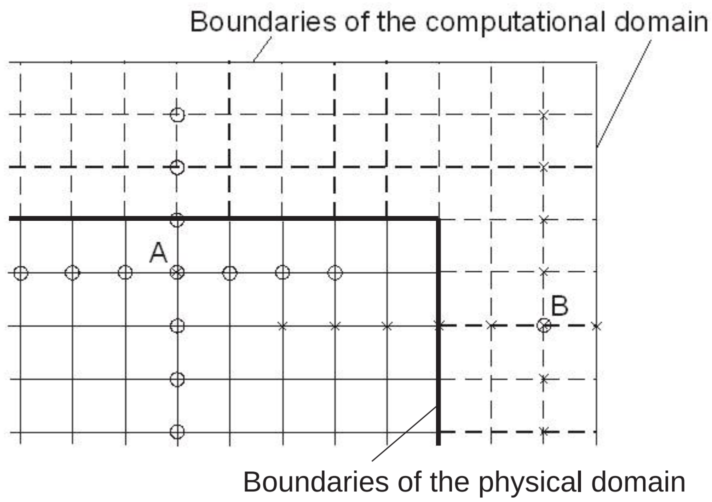
To decide whether a radiation or a outflow boundary condition should be used, one can use the radial source normal vector $\vec{n}_r$ at the boundary:

$$\vec{n}_r \cdot \vec{u}_0 \leq 0 \Rightarrow \text{ radiation boundary condition (12.2)}$$
$$\vec{n}_r \cdot \vec{u}_0 \geq 0 \Rightarrow \text{ outflow boundary condition (12.24)}$$



For 7-point-stencil schemes, three columns and/or rows of grid points very next to the boundaries of the computational domain are considered as a boundary region. In a boundary region, instead of solving the Euler equations, the radiation or outflow boundary conditions (in terms of PDEs) are solved using the same DRP scheme and the optimized multi-time level time discretization.

The two sets of the discretized equations for the boundary region and the interior region are advanced simultaneously. At some grid points, it is impossible to use symmetric spatial stencils. The optimized backward differences (7-point-stencils) are used when necessary. Point A and point B are typical grid points with symmetric spatial stencils and asymmetric spatial stencils.



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13 Nonlinear CAA

The linear wave equation does not consider certain nonlinear effects correctly:

- High amplitudes
- Interaction of the different modes of turbulence (acoustic, vortex and entropy)
- Shocks forming

Below a formulation will be derived which can handle dissipation and interaction between modes more probably, but the appearance of shocks will still be excluded. This lecture follows the considerations of Long [1]

mass conversation:

$$\frac{\partial \rho}{\partial t} + \vec{u} \cdot \left(\vec{\nabla} \rho\right) + \rho \left(\vec{\nabla} \cdot \vec{u}\right) \tag{13.1}$$

Dividing quantities in mean values and perturbations:

$$\begin{aligned} \rho &= \bar{\rho} + \rho' \\ \vec{u} &= \bar{\vec{u}} + \vec{u}' \\ p &= \bar{p} + p' \end{aligned} \tag{13.2}$$

Employing this to the mass conversation:

$$\frac{\partial \left(\bar{\rho} + \rho'\right)}{\partial t} = \left(\bar{\vec{u}} + \vec{u}'\right) \cdot \left[\vec{\nabla}(\bar{\rho} + \rho')\right] + \left(\bar{\rho} + \rho'\right) \left[\vec{\nabla} \cdot \left(\bar{\vec{u}} + \vec{u}'\right)\right] \tag{13.3}$$

It is assumed that $\bar{\rho}$ and $\bar{\vec{u}}$ are the temporal mean values and hence $\frac{\partial \bar{\rho}}{\partial t} = 0$

$$\begin{aligned} \Rightarrow \frac{\partial \rho'}{\partial t} \quad &+ \quad \underbrace{\bar{\vec{u}} \cdot (\vec{\nabla} \bar{\rho}) + \bar{\rho} (\vec{\nabla} \cdot \bar{\vec{u}})}_{=0} \\ &+ \quad \bar{\vec{u}} \cdot \left[\vec{\nabla} \rho'\right] + \vec{u}' \cdot \left[\vec{\nabla} \bar{\rho}\right] + \bar{\rho} \left[\vec{\nabla} \cdot \vec{u}'\right] + \rho' \left[\vec{\nabla} \cdot \bar{\vec{u}}\right] \\ &+ \quad \underbrace{\vec{u}' \cdot \left[\vec{\nabla} \rho'\right] + \rho' \left[\vec{\nabla} \cdot \vec{u}'\right]}_{2^{nd} \text{ order terms} \neq 0} = 0 \end{aligned} \tag{13.4}$$

The first bracket under the terms which depend only on the mean values is zero, because they itselves fulfill the mass conversation. The second brace emphasis the extra terms, which are in the nonlinear theory not negelectable.

13.1 Dimension free formulation

Equation (13.4) is applied to the characteristic factors (table 13.1), to obtain a dimensionless formulation:

quantity	characteristic factor
ρ	ρ_∞
$\vec{u}$	c_∞
t	R/c_∞
$\vec{x}$	R
p	$\rho_\infty c_\infty^2$
$\underline{\nabla}$	$1/R$

Table 13.1: characteristic factors

$$\begin{aligned}\frac{c_\infty}{R}\frac{\partial\,\rho_\infty\hat{\rho}'}{\partial\,t} &+ \frac{1}{R}\hat{\vec{u}}'c_\infty\cdot\left[\hat{\underline{\nabla}}\rho_\infty\hat{\bar{\rho}}\right]+\frac{1}{R}\rho_\infty\hat{\bar{\rho}}\left[\hat{\underline{\nabla}}\cdot c_\infty\hat{\vec{u}}'\right]\\ &+ \frac{1}{R}\hat{\bar{\vec{u}}}c_\infty\cdot\left[\hat{\underline{\nabla}}\rho_\infty\hat{\rho}\right]+\frac{1}{R}\rho_\infty\hat{\rho}'\left[\hat{\underline{\nabla}}\cdot c_\infty\hat{\bar{\vec{u}}}\right]\\ &+ \frac{1}{R}c_\infty\hat{\vec{u}}'\cdot\left[\hat{\underline{\nabla}}\rho_\infty\hat{\rho}'\right]+\frac{1}{R}\rho_\infty\hat{\rho}'\left[\hat{\underline{\nabla}}\cdot c_\infty\hat{\vec{u}}'\right]=0\end{aligned} \tag{13.5}$$

$$\begin{aligned}\Rightarrow\frac{c_\infty\rho_\infty}{R}\Bigg[\frac{\partial\,\hat{\rho}'}{\partial\,t}&+\hat{\bar{\vec{u}}}\cdot(\hat{\underline{\nabla}}\hat{\rho}')+\hat{\vec{u}}'\cdot(\hat{\underline{\nabla}}\hat{\bar{\rho}})+\hat{\bar{\rho}}(\hat{\underline{\nabla}}\cdot\hat{\vec{u}}')\\ &+\hat{\rho}'(\hat{\underline{\nabla}}\cdot\hat{\bar{\vec{u}}})+\hat{\rho}'(\hat{\underline{\nabla}}\cdot\hat{\vec{u}}')+\hat{\vec{u}}'\cdot(\hat{\underline{\nabla}}\hat{\rho}')\Bigg]=0\end{aligned} \tag{13.6}$$

Equation (13.6) is now dimensionless. It is equal to (13.4), except that the ordinary unknowns are substituted by ˆ unknowns and the constant factor $\frac{c_\infty\rho_\infty}{R}$ is put in front of the equation. If it's $\bar{\rho}=\rho_\infty$ in the whole domain, $\hat{\bar{\rho}}$ becomes 1 and all derivatives of $\hat{\bar{\rho}}$ are zero. So far it seems that applying the dimensionless form of an equation does not give many advantages.

Though, the exclusion of a constant factor yields the equation for the mass conservation is now of order 10^1 ($CFL \approx 1 = c\Delta x/\Delta t$). This is advantageous for computations, since computers have only a finite accuracy, which has its highest resolution around 1. If the equation stays in that range even after operations like multiplication and division, the truncation error is of minor importance. Beyond that the accuracy can be enhanced if $R\approx\Delta x,\Delta y,\Delta z$, and the mean values are equal to the unknowns in infinity ($\bar{\rho}=\rho_\infty$).

In addition, the momentum equation has its scale factor. Consider the momentum equation in primitive form, divided by ρ, the first term is given by:

$$\left[\frac{\partial\,\vec{u}}{\partial\,t}\right]=\frac{m}{s^2}\rightsquigarrow\frac{c_\infty^2}{R} \tag{13.7}$$

It is sufficient considering only the first term, because the entire equation must have the same dimension. Thus also the energy equation, in primitive form can be normalized, by considering only the first term:

$$\left[\frac{\partial\,p}{\partial\,t}\right]=\frac{kgm}{m^2s^3}\rightsquigarrow\frac{\rho_\infty c_\infty^3}{R} \tag{13.8}$$

A remarkable feature of the non-dimensional equations is, that the speed of sound in this modified system is equal to 1 if the characteristic factors are chosen to equal the constant mean values. Although the mean pressure is not uniform anymore, it equals $1/\gamma$. After normalization the momentum equation also has to be applied to the separation into mean value and perturbation (equation (13.2)). In the following the $\hat{}$ -mark, indicating the normalized form, will be omitted.

$$\rho\left[\frac{\partial\,\vec{u}}{\partial\,t} + \vec{u}\cdot(\underline{\nabla}\vec{u})\right] = -\underline{\nabla}p$$

$$\Rightarrow \quad (\bar{\rho}+\rho')\left[\frac{\partial\,(\overline{\vec{u}}+\vec{u}')}{\partial\,t} + (\overline{\vec{u}}+\vec{u}')\cdot\left[\underline{\nabla}(\overline{\vec{u}}+\vec{u}')\right]\right] = -\underline{\nabla}(\bar{p}+p') \tag{13.9}$$

$$\Rightarrow \quad \underbrace{\frac{\partial\,\overline{\vec{u}}}{\partial\,t} + \overline{\vec{u}}\cdot(\underline{\nabla}\overline{\vec{u}}) + \frac{\underline{\nabla}\bar{p}}{\bar{\rho}}}_{=0 \text{ (mean momentum eq.)}} - \frac{\underline{\nabla}\bar{p}(\bar{\rho}+\rho')}{(\bar{\rho}+\rho')\bar{\rho}} + \frac{\underline{\nabla}\bar{p}}{\bar{\rho}+\rho'}$$

$$+\frac{\partial\,u'}{\partial\,t} + (\overline{\vec{u}}+\vec{u}')\cdot(\underline{\nabla}\vec{u}') + \frac{\underline{\nabla}p'}{\bar{\rho}+\rho'} + \vec{u}'\cdot(\underline{\nabla}\overline{\vec{u}}) = 0 \tag{13.10}$$

with

$$\frac{\rho'}{\bar{\rho}(\bar{\rho}+\rho')} = \frac{\rho'}{\bar{\rho}\rho} = \frac{\rho}{\bar{\rho}\rho} - \frac{\bar{\rho}}{\rho\bar{\rho}} = \frac{1}{\bar{\rho}} - \frac{1}{\rho} \tag{13.11}$$

We finally get the momentum equation for the perturbed nonconservative nonlinear Euler equations (PENNE) formulation

$$\Rightarrow \frac{\partial\,u'}{\partial\,t} = -\left[(\overline{\vec{u}}+\vec{u}')\cdot(\underline{\nabla}\vec{u}') + \frac{\underline{\nabla}p'}{\bar{\rho}+\rho'} + \vec{u}'\cdot(\underline{\nabla}\overline{\vec{u}}) - \frac{\rho'}{\bar{\rho}(\bar{\rho}+\rho')}\underline{\nabla}\bar{p}\right] \tag{13.12}$$

Note: If the momentum equation is not divided by ρ the result (13.12) looks different, because the mean flow would fulfill a slightly different equation. However, it will lead to more exact results, if the mean pressure gradient is used instead of the velocity gradients.

Finally the energy equation has to be adapted to (13.2). A simplified form of it denotes

$$\frac{\partial\,p}{\partial\,t} + \vec{u}\cdot(\underline{\nabla}p) + \gamma p(\underline{\nabla}\cdot\vec{u}) = 0 \tag{13.13}$$

and with equation (13.2):

$$\frac{\partial\,\bar{p}+p'}{\partial\,t} + (\overline{\vec{u}}+\vec{u}')\cdot\left[\underline{\nabla}(\bar{p}+p')\right] + \gamma(\bar{p}+p')\left[\underline{\nabla}\cdot(\overline{\vec{u}}+\vec{u}')\right] = 0 \tag{13.14}$$

Separating the mean flow equation from the acoustic equation.

$$\frac{\partial\,\bar{p}}{\partial\,t} + \bar{u}\cdot(\underline{\nabla}\bar{p}) + \gamma\bar{p}(\underline{\nabla}\cdot\overline{\vec{u}}) = 0 \tag{13.15}$$

$$\frac{\partial\,p'}{\partial\,t} = -\left[(\overline{\vec{u}}+\vec{u}')\cdot(\underline{\nabla}p') + \gamma(\bar{p}+p')(\underline{\nabla}\cdot\vec{u}') + \vec{u}'\cdot(\underline{\nabla}\bar{p}) + \gamma p'(\underline{\nabla}\cdot\overline{\vec{u}})\right] \tag{13.16}$$

References

[1] L.N. Long. A Nonconservative Nonlinear Flowfield Splitting Method for 3-D Unsteady Fluid Dynamics. AIAA Paper 2000–1998, 2000.

Additional material

A Acoustic Noise Fundamental and Measures of Sound

Acoustic disturbance can be considered as small-amplitude perturbations to an ambient state,

$$\left(p_0,\ \rho_{0,}\ \vec{U_0}\right).$$

The ambient-field variables satisfy the governing equations.

$$p = p_0 + p'$$
$$\rho = \rho_0 + \rho'$$

etc.

where p' and ρ' represent the acoustic disturbances to the overall pressure and density field.

A.1 Definitions

Ambient state:	The medium through which sound propagates.
A homogeneous medium:	All ambient variables are independent of position.
A quiescent medium:	All ambient variables are independent of time and $\overrightarrow{U}_0 = 0$.
An isentropic medium:	Physical properties are independent of the direction in which they are measured.

From the linear wave equation,

$$\frac{\partial^2 p}{\partial t^2} = c^2 \nabla^2 p$$

we can see that at any spatial location where p is a maximum ($\frac{\partial^2 p}{\partial x^2} < 0,\ \nabla^2 p < 0$), the value of p should be accelerated forward decreasing $p\left(\frac{\partial^2 p}{\partial t^2} < 0\right)$. Otherwise $\left(\frac{\partial^2 p}{\partial t^2} = -c^2 \nabla^2 p\right)$, the acoustic pressure at the point would grow without a bound and the medium would be unstable.

A.2 Questions

(a) What is a wave?
The wave is a disturbance that travels through a medium (a gas, a liquid, or a solid)
Sonic waves: Sound waves that can be perceived by the hearing sense of a human being.

(b) How does wave travel?
<u>Longitudinal propagation:</u>
A longitudinal wave is a wave in which particles of the medium move in a direction parallel to the direction the wave moves.

(c) <u>Transverse propagation:</u>
A transverse wave is a wave in which particles of the medium move in a direction perpendicular to the direction the wave moves

A.3 Acoustic Noise Fundamentals

- Sound is a change in pressure with respect to the atmosphere
- Noise is unwanted sound
- Speed of sound in air is about 1 130 ft/sec (340 m/sec)
- Wavelength $\lambda = c/f$,

  where c = speed of sound
  f = frequency of sound (cycles per sec)

- Normal range of hearing is 20 Hz to 10 000 Hz
- Most sensitive range is 1 000 Hz to 4 000 Hz.

A.4 Measures of Sound

Root-Mean-Square Sound Pressure Most common sounds consists of a rapid, irregular series of positive-pressure disturbances (rarefaction) measured from the equilibrium pressure value. If we were to measure the mean value of the sound pressure disturbance, we would find that it would be zero because there are as much positive compression as negative rarefactions. Thus, the mean value of sound pressure is not a useful measure. We therefore look for a measure that allows the effects of the compressions to be added to the effects of the rarefactions, i.e., the *rms* sound pressure p_{rms}

(a) Squaring the value of the sound pressure disturbance at each instant of time

$$p'^2(t)$$

(b) The squared values are added and averaged over the sample time

$$\left(P^2\right)_{\mathrm{avg}} = \frac{1}{T}\int_t^{t+T} p'^2\,(t)\,dt$$

(c) The *rms* sound pressure is the square root of this time average

$$p_{\mathrm{rms}} = \sqrt{\left(P_{\mathrm{avg}}\right)^2}$$

Sound level is a logarithmic scale:

$$\text{SPL (Sound pressure level)} = 10\log\left[\frac{(p^2)_{\mathrm{rms}}}{p^2_{\mathrm{ref}}}\right] = 20\log\frac{p_{\mathrm{rms}}}{p_{\mathrm{ref}}} := L_p(\mathrm{dB})$$

where p_{ref}= reference sound pressure, which is 2×10^{-5} N/m^2 (20μPa) for airborne sound and 10^{-6} Pa for underwater sound.

The correspondence between the sound pressure level L$_p$ and p_{rms}

$$p_{\mathrm{rms}} = p_{\mathrm{ref}}10^{L_p/20}$$

Therefore, increasing L_p by 20 dB implies increasing p_{rms} by a factor of 10.

Understanding Decibels The decibel (abbreviated dB) is the unit used to measure the intensity of a sound. On the decibel scale, the smallest audible sound (near total silence) is 0 dB. A sound 10 times more powerful is 10 dB. A sound 100 times powerful than near total silence is 20 dB. A sound 1000 times powerful than near total silence is 30 dB.

The following are some common sounds and their decibel ratings:

0 dB	essentially no sound heard
15 dB	a whisper
35 dB	quite home
60 dB	normal conversation
70 dB	noisy street
90 dB	a lawn mower
110 dB	a car horn
120 dB	a rock concert
140 dB	a gunshot or firecracker

We know from our own experience that distance affects the intensity of sound, i.e. if we are far away from the sound source, the power is extremely diminished. All of the ratings above are taken while standing close to the sound source.

Adding decibels

$$\mathrm{dB}_1+\mathrm{dB}_2+\ldots+\mathrm{dB}_n = 10\log(10^{\mathrm{dB1}/10}+10^{\mathrm{dB2}/10}+\ldots+10^{\mathrm{dB}n/10})$$

Examples:

(a) Two equal decibel values (i.e. two identical fans) add to produce a 3 dB increase.
86 + 86 = 89 dB
100 + 100 = 103 dB

(b) Two decibel values differing by 6 dB add to produce a 1 dB increase
86 + 80 = 87 dB
100 + 94 = 101 dB

(c) N sources generating the same sound level are combined, the overall sound pressure level will increase by 10log(N) dB

Acoustic Intensity The acoustic intensity I is defined as

$$I = \frac{1}{T}\int_0^T p\,\vec{u}\cdot\vec{n}\,dt \text{ (Watt /m}^2\text{)}$$

It is the time-averaged rate of energy transmission through a unit area normal to the direction of propagation. It is the product of sound pressure and acoustic particle velocity.

For a plane harmonic wave

$$I = \pm\frac{P^2}{2\rho_0 c}$$

where P is the peak acoustic pressure amplitude.

Acoustic power Acoustic energy delivered per unit time (Watts), i.e. the energy transmission through an area normal to the direction of sound propagation.

Directivity Directivity is a measure of the directional characteristic of a sound source. It is the ratio of the sound intensity produced by a sound source on a specific axis to that of a point source that is producing the same acoustic power.

$$D = \frac{I_{ax}(r)}{I_s(r)}$$

Noise Analysis

- Microphone - Measures pressure fluctuations
- Analyzer - Takes amplified pressure fluctuations and processes the information into useful information.
- Octave Band Analysis

Octave Band Because most sound are complex, fluctuating in amplitude and frequency content, the relationships between sound energy level and frequency are required for meaningful analysis (data plotted in this way is called *sound spectrum*).

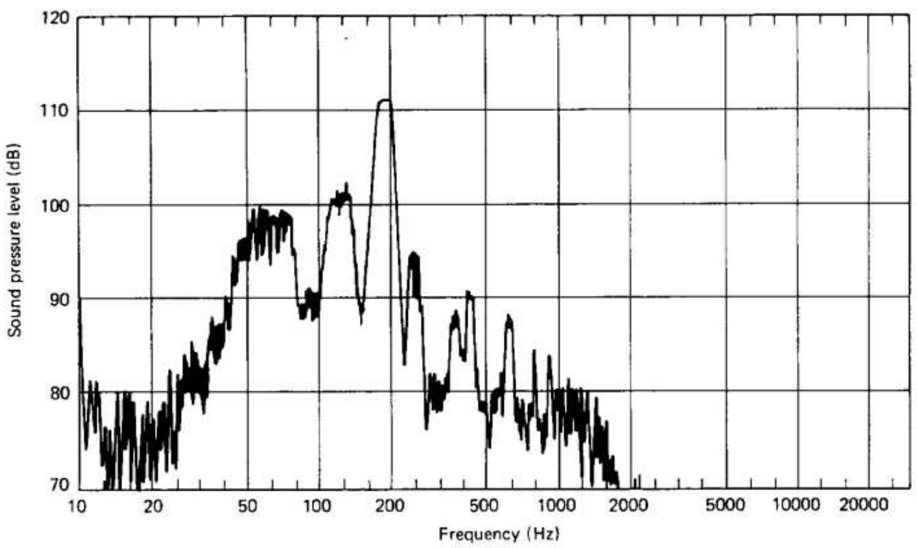

Figure A.1: Sound Spectrum of an Air-Compressor

For most engineering applications themain concern is in the frequency range from 20 to 20,000 Hz. Although it is possible to analyze a source on a frequency by frequency basis, this is both impractical and time-consuming. For this reason, a scale of *octave bands* and *one-third octave bands* has been developed. Each band covers a specific range of frequencies and excludes all others. The word "octave" is borrowed from musical nomenclature where it refers to a span of eight notes, i.e. $\overline{do}$ to $\overline{do}$. The ratio of the frequency of the highest note to the lowest note in an octave is 2:1.

If f_n is the lower cut-off frequency and f_{n+1} is the upper cut-off frequency, the ratio of band limits is given by

$$\frac{f_{n+1}}{f_n} = 2^k \qquad \text{with} \begin{cases} k = 1 & \text{for full octave bands} \\ k = \frac{1}{3} & \text{for one-third octave bands} \end{cases}$$

An octave has a center frequency that is $\sqrt{2}$ times the lower cut-off frequency and has an upper cut-off frequency that is twice the lower cut-off frequency. Therefore,

$$\begin{array}{lll} f_2 &= 2\,f_1 & \text{upper cut-off frequency} \\ f_0 &= \sqrt{2}\,f_1 & \text{center frequency} \\ f_1 &= f_0/\sqrt{2} & \text{lower cut-off frequency} \\ bw &= f_2 - f_1 & \text{band width} \end{array}$$

Comparison of 1-octave and ⅓-octave bands

1 Octave			⅓ Octave		
Lower cutoff frequency (Hz)	Center frequency (Hz)	Upper cutoff frequency (Hz)	Lower cutoff frequency (Hz)	Center frequency (Hz)	Upper cutoff frequency (Hz)
11	16	22	14.1	16	17.8
			17.8	20	22.4
			22.4	25	28.2
22	31.5	44	28.2	31.5	35.5
			35.5	40	44.7
			44.7	50	56.2
44	63	88	56.2	63	70.8
			70.8	80	89.1
			89.1	100	112
88	125	177	112	125	141
			141	160	178
			178	200	224
177	250	355	224	250	282
			282	315	355
			355	400	447
355	500	710	447	500	562
			562	630	708
			708	800	891
710	1,000	1,420	891	1,000	1,122
			1,122	1,250	1,413
			1,413	1,600	1,778
1,420	2,000	2,840	1,778	2,000	2,239
			2,239	2,500	2,818
			2,818	3,150	3,548
2,840	4,000	5,680	3,548	4,000	4,467
			4,467	5,000	5,623
			5,623	6,300	7,079
5,680	8,000	11,360	7,079	8,000	8,913
			8,913	10,000	11,220
			11,220	12,220	14,130
11,360	16,000	22,720	14,130	16,000	17,780
			17,780	20,000	22,390

A.5 Air Mover Selection Hints

- Run as slow as possible
- Pick a fan that will operate at peak efficiency
- Be aware of blade passing frequency and harmonics
- Blade Passing frequency = N·Z/60 (Hz)

  where N = # of blades
  Z = rotational frequency in RPM

- Fundamental noise frequency must not equal mounting structure natural frequency
- Limit line of sight to prop / rotor
- Acoustic dampening foam
- Filters can dampen noise
- Mechanical isolation
- Control air velocity in system. The lower the better.
- Keep sharp objects away from prop / rotor. This will create another fundamental frequency
- Put fan as far away from noise critical areas as possible

Noise Basics There are three basic sources of noise in an air mover: aerodynamic, mechanics and motor. Aerodynamic noise is comprise of three basic components:

blade noise
turbulence
vortex shedding noise

Of these three, the blade-passing tone generally dominates the noise. The blade-passing tone results from the air momentum which occur every time a blade passes a given point. The number of times such momentums occur per second determines the fundamental blade-passing frequency. This is given by:

$$f_{\text{fund}} = \frac{N \cdot Z}{60} \text{ Hz, where } \begin{array}{l} \text{Z = number of blades} \\ \text{N = rotation speed in RPM} \end{array}$$

As the speed or the number of blades increases, the fundamental blade-passing frequency increases.

Small scaled turbulence is present in almost any environment. As this passes through a fan or blower, it produces a fluctuating lift on the blades. This produces a fluctuating pressure in the air and hence noise. The frequency characteristics of such noise is broad-band and the intensity generally low, unless the turbulence intensity is caused by sharp objects or obstructions adjacent to the inlet of the fan or blower.

Mechanical noise in fans and blowers can be caused by either vibration or misalign and worn bearings. Imbalances in rotary components are the primary cause of vibration which ultimately results in structural noise and at times failure of the component due to resonance. The frequency of such noise depends on the rotational speed. The fundamental frequency due to rotary imbalance is given by

$$f_n = \frac{N}{60}\text{, where } N = \text{rotation speed (RPM)}$$

B Aliasing

For a fixed, mesh size (Δx), the resolved wavenumber range is $-\pi \le \alpha\Delta x \le \pi$. For ultra short waves with wavelengths less then $2\Delta x$ (or $\alpha\Delta x\pi$), the waves are unresolved in a finite difference computation. They are aliased into different wavenumber.

Consider the problem

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0$$

with initial condition

$$u(t=0) = e^{-\ln 2\left(\frac{x}{b}\right)^2} \cos\left(\alpha_0 x\right)$$

For convenience of analysis, the initial condition is modified to a complex expression (the real part of the initial condition is used to obtain the solution)

$$u(t=0) = e^{-\ln 2\left(\frac{x}{b}\right)^2 + i\alpha_0 x}$$

The Fourier transform of the initial condition is

$$\tilde{u}(t=0) = \frac{b}{2\sqrt{\pi \ln 2}} e^{-\ln 2\left[\frac{(\alpha-\alpha_0)b}{2\ln 2}\right]^2}$$

In the wavenumber space, the initial Gaussian pulse has a half-width of $\frac{2\ln 2}{b}$. The initial condition represents waves concentrating around a wavenumber of $\alpha = \alpha_0$. The discretized expression for the initial condition is

$$f_l = u_l = e^{-\ln 2\left(\frac{l\Delta x}{b}\right)^2} e^{i\alpha_0 \Delta x l}$$

Assuming $\pi < \alpha_0\Delta x < 2\pi$ (α_0 unresolved wavenumber) we can write

$$\boxed{\alpha_0 \Delta x = \pi + \delta}$$

and therefore

$$e^{i\alpha_0\Delta x l} = e^{i[2\pi + (\delta - \pi)]l} = e^{i(\delta-\pi)l}$$

The discretized initial condition becomes

$$f_l = u_l = e^{-\ln 2\left(\frac{l\Delta x}{b}\right)^2 + i\,(\delta-\pi)l}$$

In terms of a continuous variable x, the initial condition is

$$f\left(x\right) = u\left(x\right) = e^{-\ln 2\left(\frac{x}{b}\right)^2 + i\frac{(\delta-\pi)x}{\Delta x}}$$

This represents waves concentrating around a wavenumber $\alpha = \frac{\delta-\pi}{\Delta x} = \left(\alpha_0 - \frac{2\pi}{\Delta\pi}\right)$. These are waves lying within the resolved wavenumber range. As we have shown that unresolved wavenumber $\alpha_0\Delta x$ becomes resolved wavenumber $\alpha\Delta x$ due to the discretization. We define the aliased wavenumber as Aliased wavenumber

$$\boxed{\alpha = \alpha_0 - \frac{2\pi}{\Delta x}} \quad \text{with} \quad \begin{array}{ll} \alpha: & \text{aliased wavenumber} \\ \alpha_0: & \text{wavenumber} \end{array}$$

B.1 Numerical Examples

(a) $\alpha_0\Delta x = 3.9$ $\quad b = 20\Delta x$ $\quad \alpha\Delta x = 3.9 - 2\pi = -2.383$
For this aliased wavenumber, the group velocity is negative

(b) $\alpha_0\Delta x = 5.48$ $\quad b = 20\Delta x$ $\quad \alpha\Delta x = 5.48 - 2\pi = -0.803$
For this aliased wavenumber, the group velocity is nearly equal to 1.0.

C High-Order Optimized Upwind Schemes

Consider the one-dimensional scalar model wave equation

$$\frac{\partial u}{\partial t} + C\frac{\partial u}{\partial x} = 0$$

where C is the speed of wave propagation.

Without the loss of generality, we assume that $C > 0$. The approximation of the first order spatial derivative $\partial u/\partial x$ by a finite difference on an uniform grid of spacing Δx is given by

$$\left(\frac{\partial u}{\partial x}\right)_l \approx \frac{1}{\Delta x}\sum_{j=-N}^{M} a_j u_{l+j}$$

By performing the Fourier transform on the continuous form of the above equation, we define

$$\begin{aligned} \bar{\alpha} &= -\frac{i}{\Delta x}\sum_{j=-N}^{M} a_j e^{i\alpha j\Delta x} \\ \Rightarrow \quad \overline{\alpha}\Delta x &= -i\sum_{j=-N}^{M} a_j e^{i\alpha j\Delta x} \end{aligned}$$

Let $\bar{\beta} = \overline{\alpha}\Delta x$, $\bar{\beta} = \beta_r + i\beta_i$ and $\beta = \alpha\Delta x$. The integrated error E is defined as

$$E = \int_0^{\beta_0} \left|\overline{\beta}_r - \beta\right|^2 d\beta + \lambda \int_0^{\beta_0} \left|\overline{\beta}_i + \operatorname{sgn}(C)\exp\left[-\ln 2\left(\frac{\beta-\pi}{\sigma}\right)^2\right]\right|^2 d\beta$$

where β_0 is a predetermined number (same like the central DRP schemes discussed before) which gives the optimized range of wavenumber. The parameter λ is a weighting coefficient and the parameter τ adjusts the width of the Gaussian function.

$$\text{Since } \overline{\beta}_r = \mathfrak{Re}\left\{\overline{\beta}\right\} = \sum_{j=-N}^{M} a_j \sin(j\beta)$$

$$\overline{\beta}_i = \mathfrak{Im}\left\{\overline{\beta}\right\} = -\sum_{j=-N}^{M} a_j \cos(j\beta)$$

$$\begin{aligned} E = &\int_0^{\beta_0}\left[\sum_{j=-N}^{M} a_j \sin(j\beta) - \beta\right]^2 d\beta \\ &+ \lambda\int_0^{\beta_0}\left[-\sum_{j=-N}^{M} a_j\cos(j\beta) + \exp\left[-\ln 2\left(\frac{\beta-\pi}{\sigma}\right)^2\right]\right]^2 d\beta \end{aligned} \quad \text{(C.1)}$$

The necessary conditions that E is a minimum for all the free coefficients are

$$\frac{\partial E}{\partial a_l} = 0, \qquad l = -N, -N+1, \ldots, M-P-1$$

where P is the order of accuracy of the finite difference scheme and $P \leq M + N$

$$\left(\text{e.g.} \quad M = 2, \quad N = 4, \quad P = 4, \quad \frac{\partial E}{\partial a_l} = 0, \quad l = -4, -3\right)$$

The first term in equation (C.1) minimizes the distance between $\overline{\beta}_r$ and β in the form of L_2 norm. The second term, instead of minimizing the distance between $\overline{\beta}_i$ and 0, minimizes the distance between $\overline{\beta}_i$ and a Gaussian function in the form of L_2 norm. The Gaussian function is almost zero when the value of β is far from π. The Gaussian term is chosen such that the imaginary part of the effective wavenumber $\overline{\beta}_i$ is very close to zero for waves with wavenumbers within a certain range. The second term also allows controls over short wave or high frequency damping by adjusting the control parameter σ.

As it is in a traditional finite difference scheme, the upwind scheme uses more stencils upwind (from the left) than downwind since the wave is assumed to propagate to the right ($C > 0$). If the wave is propagation to the left ($C < 0$), the upwind biased scheme uses more stencil points from the right than those from the left

$$\text{(e.g.} \quad C > 0: \quad M = 2, \quad N = 4; \quad C < 0: \quad M = 4, \quad N = 2)$$

The coefficients of the DRP scheme for $C < 0$ can be derived directly from its corresponding counterpart for $C > 0$ if the same number of stencil points and the same fashion of bias are used.

The effective wavenumber $\overline{\alpha}$ of the scheme with coefficients a_j^{MN} is the complex conjugate of that of the scheme with coefficients a_j^{NM}. The opposite sign in the imaginary part of the effective wavenumber $\overline{\alpha}$ of the two schemes is needed to ensure the stability of the schemes for waves propagating in two contrary directions. That is

$$\begin{array}{rcl} a_j^{NM} & = & -a_{-j}^{MN} \\ \text{e.g.} \quad N = 4, \quad M = 2, & & j \text{ is from } -N \text{ to } M \\ a_{-4}^{42} & = & -a_4^{24} \\ a_{-3}^{42} & = & -a_3^{24} \\ a_{-2}^{42} & = & -a_2^{24} \\ a_{-1}^{42} & = & -a_1^{24} \\ a_0^{42} & = & -a_0^{24} \\ a_1^{42} & = & -a_{-1}^{24} \\ a_2^{42} & = & -a_{-2}^{24} \\ \uparrow & & \uparrow \\ C > 0 & & C < 0 \end{array}$$

C.1 Procedures for finding coefficients a_j

$$\left.\begin{array}{l} \sum\limits_{j=-N}^{M} a_j = 0 \\ \sum\limits_{j=-N}^{M} a_j j = 1 \\ \sum\limits_{j=-N}^{M} a_j j^k = 0 \end{array}\right\} \quad (P+1) \quad \text{(number of } \textit{equations}\text{)}$$

where $k = 2, 3, \ldots, P$. Together with the minimization condition

$$\frac{\partial E}{\partial a_l} = 0, \qquad l = -N, -N+1, \ldots, M-P-1$$

$\uparrow$

$(M + N - P)$ (number of equations)

Therefore we have the total $M + N + 1$ unknowns and $M + N + 1$ equations (one linear system).

The partial derivatives of a_j over a_l appeared in $\frac{\partial E}{\partial a_l}$ can be obtained by solving the following linear system.

$$\frac{\partial a_j}{\partial a_l} = \delta_{jl} \qquad \text{for } j = -N, -N+1, \ldots, M-P-1,$$

and

$$\sum_{j=-N}^{M} j^{\kappa} \frac{\partial a_j}{\partial a_l} = 0 \qquad \text{for } \kappa = 0, 1, \ldots, P.$$

For $M = N$ and $a_{-j} = -a_j$, the finite difference scheme becomes a central different scheme

$$\mathfrak{Im}\left\{\overline{\beta}\right\} = -\sum_{j=-N}^{M} a_j \cos(j\beta) = a_0 - \sum_{j=1}^{M} (a_j \cos(j\beta) + a_j \cos(j\beta)) = a_0$$

Since $\sum\limits_{j=-N}^{M} a_j = 0$ (Taylor series expansion), it implies $a_0 = 0$ for central scheme.

For $\lambda = 0.0374$ and $\tau = 0.2675\pi$, the coefficients a_j are given in tables 1 & 2.0.

C.2 Some General Note on Upwind Schemes

Consider again the "wave equation".

$$\frac{\partial u}{\partial t} + C\frac{\partial u}{\partial x} = 0$$

Now we do not make any assumptions as to the sign of C. We can rewrite the above equation as

$$\frac{\partial u}{\partial t} + \left(C^+ + C^-\right)\frac{\partial u}{\partial x} = 0 \quad \text{where } C^\pm = \frac{C \pm |C|}{2} \quad \begin{cases} C \geq 0: & C^+ = C,\ C^- = 0 \\ C \leq 0: & C^+ = 0,\ C^- = C \end{cases}$$

Now for the C^+ (≥ 0) term we can safely use backward difference and for C^- (≤ 0) forward difference. This is the basic concept behind upwind methods, that is, some decomposition or splitting of the fluxes into terms which have positive and negative characteristic speeds so that appropriate differencing schemes can be chosen.

Flux-Vector Splitting The Euler equations form a hyperbolic system of partial differential equations. Many aspects of numerical methods for such a system can be understood by studying a one-dimensional constant-coefficient linear system of the form

$$\frac{\partial \vec{u}}{\partial t} + A\frac{\partial \vec{u}}{\partial x} = 0$$

where $\vec{u} = \vec{u}\,(x, t)$ is a vector of length m and A is a real $m \times m$ matrix.

For conservation laws, this equation can be written as

$$\frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{f}}{\partial x} = 0 \tag{C.2}$$

where $\vec{f}$ is the flux vector and $A = \frac{\partial \vec{f}}{\partial \vec{u}}$ is the flux Jacobian matrix. The entities in the flux Jacobian are

$$a_{ij} = \frac{\partial f_i}{\partial u_j}$$

The matrix A can be diagonalized

$$\Lambda = X^{-1}AX$$

where Λ is a diagonal matrix containing the eigenvalues of A, and X is the matrix of right eigenvectors. Equation (C.2) then can be written as

$$\frac{\partial X^{-1}\vec{u}}{\partial t} + \frac{\partial \overbrace{X^{-1}AX}^{\Lambda}\,X^{-1}\vec{u}}{\partial x} = 0$$

With $\vec{w} = X^{-1}\vec{u}$, we have

$$\boxed{\frac{\partial \vec{w}}{\partial t} + \Lambda\frac{\partial \vec{w}}{\partial x} = 0} \tag{C.3}$$

Equation (C.3) can be decoupled into m scalar equations of the form

$$\frac{\partial w_i}{\partial t} + \lambda_i\frac{\partial w_i}{\partial x} = 0$$

The elements of $\vec{w}$ are known as characteristic variables. Each characteristics variables satisfies the linear "wave" convection equation with the speed given by the corresponding eigenvalue of A.

Now let us split the matrix of eigenvalues Λ into two components such that

$$\Lambda = \Lambda^+ + \Lambda^- \qquad \text{where } \Lambda^\pm = \frac{\Lambda \pm |\Lambda|}{2}$$

With these definitions, Λ^+ contains the positive eigenvalues and Λ^- contains the negative eigenvalues. Now equation (C.3) can be rewritten as

$$\frac{\partial \vec{w}}{\partial t} + \Lambda\frac{\partial \vec{w}}{\partial x} = \frac{\partial \vec{w}}{\partial t} + \Lambda^+\frac{\partial \vec{w}}{\partial x} + \Lambda^-\frac{\partial \vec{w}}{\partial x} = 0$$

We can then use backward differencing for the term $\Lambda^+\frac{\partial \vec{w}}{\partial x}$ and forward differencing for the term $\Lambda^-\frac{\partial \vec{w}}{\partial x}$. Pre-multiplying by X, the matrix of right eigenvector of $A(Ax_R = \lambda_R x)$, and inserting the product $X^{-1}X$ in the spatial terms, we have

$$\frac{\partial X\vec{w}}{\partial t} + \frac{\partial X\Lambda^+X^{-1}X\vec{w}}{\partial x} + \frac{\partial X\Lambda^-X^{-1}X\vec{w}}{\partial x} = 0$$

Define

$$A^\pm = X\Lambda^\pm X^{-1}$$

and recall that $\vec{u} = X\vec{w}$, we obtain

$$\frac{\partial \vec{u}}{\partial t} + \frac{\partial A^+\vec{u}}{\partial x} + \frac{\partial A^-\vec{u}}{\partial x} = 0$$

The split flux vectors are defined as

$$\begin{aligned} \vec{f}^\pm &= A^\pm\vec{u} \\ \vec{f} &= \vec{f}^+ + \vec{f}^- \\ 0 &= \frac{\partial \vec{u}}{\partial t} + \frac{\partial \vec{f}^+}{\partial x} + \frac{\partial \vec{f}^-}{\partial x} \end{aligned}$$

where $\vec{f} = A\vec{u}$ for the Euler equations.

Thus, by applying backward differences to the $\vec{f}^+$ term and forward differences to the $\vec{f}^-$ term, we are in effect solving the characteristic equations in the desired manner. This approach is known as flux-vector splitting.

For the linear Euler equations (or the acoustic field equations), we then have

$$\frac{\partial \vec{q}}{\partial t} + A^+\frac{\partial \vec{q}}{\partial x} + A^-\frac{\partial \vec{q}}{\partial x} + B^+\frac{\partial \vec{q}}{\partial y} + B^-\frac{\partial \vec{q}}{\partial y} + C^+p\frac{\vec{q}}{z} + C^-\frac{\partial \vec{q}}{\partial z} = \vec{S}$$

where

$$\vec{q} = \begin{bmatrix} \rho' \\ u' \\ v' \\ w' \\ p' \end{bmatrix}$$

We need to apply backward differences to the terms with A^+, B^+ and C^+ and forward differences to the terms with A^-, B^- and C^-.

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